Classical Inhomogeneous Heisenberg Spin Chains: Some Exact Low Energy Solutions and Statistics

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Abstract

We derive a continuum approximation for the low-energy dynamics of classical Heisenberg spin chains with generally inhomogeneous coupling interactions between nearest neighbour spins, $J = J(x)$. We show that the known ferromagnetic and antiferromagnetic homogeneous cases can be obtained as particular cases of this formulation and report several new results: a) To lowest order in an energy parameter, ϵ, we solve for chains with blocks of three spins, where within the block the couplings are (J,J,-J); b) We derive explicit equations for the spin configurations along the chain when $J(x)$ varies slowly along the chain; c) We discuss the statistics of the spins when $J(x)$ is fluctuating either randomly or with given correlations. (This is plenty and much of it is conditional upon our progress. RB)

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1. Introduction

1. Background on spin chains.
2. Recent results

Ok, done with the intro..

2. Introduction

Consider a chain of classical Heisenberg spins with the hamiltonian

$$H = - \sum_{<i,j>=1}^{N} J_{ij} \mathbf{S}_i \mathbf{S}_j \ , \tag{2.1}$$ “Ai

where $\mathbf{S}_i$ is the spin vector at the i-th posittion along the chain and J_{ij} is the coupling interaction between the spins at positions i and j. In the following it will be assumed that this coupling constant vanishes for spins that are not nearest neighbours, which restricts the sum and is represented by the angular brackets that enclose the summation variables. For our purpose here it is convenient to rewrite the coupling constants in the form

$$J_{ij} = J \xi_i \xi_j \ , \tag{2.2}$$ “Aii

where J is a positive constant that does not depend on position. The Hamiltonian can be rewritten in the form

$$H = -J \sum_{<i,j>=1}^{N} \boldsymbol{\tau}_i \boldsymbol{\tau}_j \ , \tag{2.3}$$ “Aiii

where we have defined

$$\boldsymbol{\tau}_i \equiv \xi_i \mathbf{S}_i \ .$$

A few cases that exemplify the usefulness of such a description would be $\xi_i = 1$ for all i (the homogeneous ferromagnetic chain), $\xi_{i+1} = -\xi_i$ for all i (the homogeneous antiferromagnetic chain), $\xi_i = \pm 1$ in either a random or any other fashion (disordered chains), and $\xi_i = f(i)$, with $f(i)$ some continuous function of i (smoothly disordered chains). For the initial stages of the analysis, we shall restrict the absolute value of ξ to unity. But this is not a serious limitation and will be lifted in some of the cases that we shall discuss later. The new Hamiltonian (2.3) describes a homogeneous ferromagnetic Heisenberg spin chain, with $\boldsymbol{\tau}$ normalised exactly as $\mathbf{S}$, $|\boldsymbol{\tau}| = |\mathbf{S}| = S$.

The transformed Hamiltonian (2.3) describes a homogeneous spin chain of ferromagnetic spins whose ground state (GS) is trivial – all $\boldsymbol{\tau}_i$ point in the same direction. Knowing the GS is essential to a reasonable construction of a low-energy approximation of the dynamics. Indeed, previous analyses for the ferromagnetic [1]and the antiferromagnetic [2]chains were able to construct equations for the dynamics in this regime exactly because the GS is known. Reciprocally, the main problem in solving for more complicated configurations of the ξ_i is that their GS is not known. In view of this difficulty the advantage of the simple transformation to the form (2.3) becomes apparent; its GS is known regardless of the values of the ξ_i.

The low-energy continuum approximation is constructed now as follows: We first derive the general equations of motion for the vectors $\boldsymbol{\tau}_i$.

$$
\begin{aligned}
\dot{\boldsymbol{\tau}}_l^{(\delta)} = \left\{\boldsymbol{\tau}_l^{(\delta)}, H\right\} &= \sum_{j=1}^{N} \sum_{\alpha,\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \frac{\partial \boldsymbol{\tau}_l^{(\delta)}}{\partial \mathbf{S}_j^{(\alpha)}} \frac{\partial H}{\partial \mathbf{S}_j^{(\beta)}} \mathbf{S}_j^{(\gamma)} \\
&= -J \sum_{\beta,\gamma=1}^{3} \epsilon_{\delta\beta\gamma} \xi_l \mathbf{S}_l^{(\gamma)} \left[\xi_l \xi_{l+1} \mathbf{S}_{l+1}^{(\beta)} + \xi_{l-1} \xi_l \mathbf{S}_{\underset{l-1}{(\beta)}}\right] \\
&= -J \sum_{\beta,\gamma=1}^{3} \epsilon_{\delta\beta\gamma} \xi_l \boldsymbol{\tau}_l^{(\gamma)} \left(\boldsymbol{\tau}_{l+1}^{(\beta)} + \boldsymbol{\tau}_{l-1}^{(\beta)}\right) \ .
\end{aligned}
\qquad (2.4) \quad \text{“Aiv}
$$

or

$$\dot{\boldsymbol{\tau}}_l = J\xi_l \boldsymbol{\tau}_l \times (\boldsymbol{\tau}_{l+1} + \boldsymbol{\tau}_{l-1}) \ . \tag{2.5}$$ “Av

In the above relations α, β and γ denote components of the vector $\boldsymbol{\tau}$ and $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric Levi-Civita (spelling?) tensor. Since the GS is such that all the $\boldsymbol{\tau}_l$ are pointing at the same (say the z-) direction it is safe to presume that at low energies $\boldsymbol{\tau}_{l+1}$ deviates only slightly from from $\boldsymbol{\tau}_l$. We therefore quantify this deviation and, by going to the continuum limit, express the above difference equation as a differential one.

3. Homogeneous ferromagnetic and antiferromagnetic spin chains

Let us first make contact with exisiting analyses and address the homogeneous ferromagnetic and antiferromagnetic cases within one treatment. For this purpose, it is convenient to choose two-spin blocks for the expansion, $\boldsymbol{\tau}_{e,n} = \boldsymbol{\tau}_l(x)$ and $\boldsymbol{\tau}_{o,n} = \boldsymbol{\tau}_{l-1}(x-a)$. The index $n = 1, ..., N/2$ denotes a particular block of two spins (not a site), and the indices e and o specify the even ($l = 2n$) and odd ($l = 2n-1$) position within the n-th block. We pass to the continuum limit in Eqs. (2.5) by

$$\boldsymbol{\tau}_{\alpha,n} - \boldsymbol{\tau}_{\alpha,n-1} \to 2a\partial_x \boldsymbol{\tau}_{\alpha,n} \quad ; \quad \alpha = e,\ o\ . \tag{3.1}$$ “Avi

This gives, after a straightforward manipulation,

$$\begin{aligned} \dot{\boldsymbol{\tau}}_e &= 2J\xi_e \boldsymbol{\tau}_e \left(\boldsymbol{\tau}_o + a\partial_x \boldsymbol{\tau}_o + a^2\partial_{xx}\boldsymbol{\tau}_o\right) \\ \dot{\boldsymbol{\tau}}_o &= 2J\xi_o \boldsymbol{\tau}_e \left(\boldsymbol{\tau}_e - a\partial_x \boldsymbol{\tau}_e + a^2\partial_{xx}\boldsymbol{\tau}_e\right)\ . \end{aligned} \tag{3.2}$$ “Avii

We now define two orthonormal vectors via

$$\begin{aligned} 2S\epsilon\mathbf{M} &= \boldsymbol{\tau}_e - \boldsymbol{\tau}_o \\ 2S\chi\mathbf{N} &= \boldsymbol{\tau}_e + \boldsymbol{\tau}_o\ , \end{aligned} \tag{3.3}$$ “Aviii

where $\chi = \sqrt{1-\epsilon^2}$ and ϵ is a small parameter to be defined below. Since at low energies the vector $\boldsymbol{\tau}$ changes very slowly along the chain we define [2]

$$2\epsilon^2(x) \equiv \left(1 - \boldsymbol{\tau}_e \cdot \boldsymbol{\tau}_o / S^2\right) \ . \tag{3.4}$$ “Aix

In the following we assume that $\epsilon(x) = \epsilon$ for all x and regard it as a small energy parameter that can be used for expansion.

Eqs. (3.2) can be now rewritten in terms of **M** and **N**, which gives for the vector **N**

$$\begin{aligned} 2\dot{\mathbf{N}} =& \xi^- \chi \mathbf{N} \times \partial_x \mathbf{N} + \xi^+ \chi \mathbf{N} \times \partial_{xx} \mathbf{N} + \\ & \epsilon\left[2\xi^- \mathbf{M} \times \mathbf{N} + \xi^+ \partial_x (\mathbf{M} \times \mathbf{N}) + \xi^- (\partial_{xx} \mathbf{M} \times \mathbf{N} + \mathbf{M} \times \partial_{xx} \mathbf{N})\right] + \mathcal{O}(\epsilon^2) \ , \end{aligned} \tag{3.5}$$ “Ax

where $\xi^\pm \equiv \xi_e \pm \xi_o$, time was rescaled by $t \to t' = 2JSt$, and length were rescaled by $x \to x' = x/a$. Since the diference $\boldsymbol{\tau}_e - \boldsymbol{\tau}_o$ is very small in the low energy regime the vector **M** is not particularly interesting for the purpose of this discussion. For the homogeneous ferromagnetic chain the value of ξ is constant along the chain, $\xi = 1$, which implies $\xi^- = 0$ and $\xi^+ = 2$. Therefore,

$$\dot{\mathbf{N}}_F = \mathbf{N}_F \times \partial_{xx} \mathbf{N}_F + \mathcal{O}(\epsilon) \ , \tag{3.6}$$ “Axi

which is the well-known form leading to soliton-type solutions [1]. For the homogeneous antiferromagnetic chain the value of ξ alternates along the chain, $\xi_e = -\xi_o = 1$, which implies $\xi^- = 2$ and $\xi^+ = 0$. This leads to

$$\dot{\mathbf{N}}_{AF} = \mathbf{N}_{AF} \times \partial_x \mathbf{N}_{AF} + \mathcal{O}(\epsilon) \ , \tag{3.7}$$ “Axii

which, was shown to be equivalent to the Belavin-Polyakov equation [3][2], and analysed in [4].

4. Periodic forms of ξ

Highlights

Here we give a general treatment for spin chains with a periodic form of ξ. Suppose the periodicity is over L sites, namely, $\xi_l = \xi_{l+L}$. Then we partition the spin chain into N/L blocks, each containing L spins. We form a new set of L vectors that replace $\boldsymbol{\tau}_{\alpha,n}$ ($\alpha = 1, ..., L$ and $n = 1, ..., N/L$) in the same way that $\mathbf{N}$ and $\mathbf{M}$ did for $L = 2$.

1. We identify a small energy parameter.
2. We write down the L discrete equations of motion for these vectors.
3. We expand with this parameter and pass to the continuum.
4. We solve the equations to zero order in that parameter.
5. **Example**: The triadic chain, $L = 3$, $\xi_{\alpha=1,2,3} = 1,\ 1,\ -1$.

First write down the equations of motion for $\boldsymbol{\tau}_{\alpha,n}(x-a) = \boldsymbol{\tau}_{l-1}$, $\boldsymbol{\tau}_{\beta,n}(x) = \boldsymbol{\tau}_l$ and $\boldsymbol{\tau}_{\gamma,n}(x+a) = \boldsymbol{\tau}_{l+1}$.

$$\begin{aligned}
\dot{\boldsymbol{\tau}}_{\alpha,n} &= J\xi_\alpha \boldsymbol{\tau}_{\alpha,n} \times (\boldsymbol{\tau}_{\gamma,n-1} + \boldsymbol{\tau}_{\beta,n}) \\
\dot{\boldsymbol{\tau}}_{\beta,n} &= J\xi_\beta \boldsymbol{\tau}_{\beta,n} \times (\boldsymbol{\tau}_{\alpha,n} + \boldsymbol{\tau}_{\gamma,n}) \\
\dot{\boldsymbol{\tau}}_{\gamma,n} &= J\xi_\gamma \boldsymbol{\tau}_{\gamma,n} \times (\boldsymbol{\tau}_{\beta,n} + \boldsymbol{\tau}_{\alpha,n+1})
\end{aligned} \qquad (4.1) \quad \text{“Pi}$$

.

We define as the small energy parameter the quantity (negotiable)

$$\epsilon^2 = 1 - (\boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta + \boldsymbol{\tau}_\beta \cdot \boldsymbol{\tau}_\gamma + \boldsymbol{\tau}_\gamma \cdot \boldsymbol{\tau}_\alpha)/(3S^2)\ . \qquad (4.2) \quad \text{“Pii}$$

We define the following three vectors:

$$\begin{aligned}
2S\chi\mathbf{N}_T &= \boldsymbol{\tau}_\alpha + \boldsymbol{\tau}_\gamma \\
2S\epsilon\mathbf{M}_T &= \boldsymbol{\tau}_\gamma - \boldsymbol{\tau}_\alpha \\
\boldsymbol{\tau}_T &= \boldsymbol{\tau}_\beta\ .
\end{aligned} \qquad (4.3) \quad \text{“Piii}$$

Passing to the continuum, rewriting the resultant equations in terms of the three new vectors, and using $\xi_\alpha = \xi_\beta = -\xi_\gamma = 1$, we obtain for the two vectors $\mathbf{N}_T$ and $\boldsymbol{\tau}_T$, to lowest order in ϵ, the following equations

$$\dot{\mathbf{N}}_T = -\mathbf{N}_T \times \partial_x \mathbf{N}_T$$
$$\dot{\boldsymbol{\tau}}_T = \boldsymbol{\tau}_T \times \mathbf{N}_T \ . \qquad (4.4)$$

"Piv

From (4.4) we observe that the vector $-\mathbf{N}_T$ satsifies the same Belavin-Polyakov equation as in the homogeneous antiferromagnetic chain, Eq. (3.7). Therefore it can be solved exactly leading to the instantons and multi-twist solutions found in [4]. Once the solution for $\mathbf{N}_T$ is given, we can solve for $\boldsymbol{\tau}_T$ from the second of Eqs. (4.4).

5. Inhomogeneous chains

Here we discuss disordered chains. We analyse two classes of disorder: A. The coupling J varies slowly as a function of position along the chain; B. The coupling J is a random variable that can vary sharply between two neighbouring bonds along the chain. In the first case we exploit the slow variation of J to locally approximate it to linear order and derive the new equation of motion. The second case has to be treated statistically and reduces to a differential equation with stochastic parameters. Approaches for treating the equation of motion are discussed.

A. *Smoothly disordered chains*

Let the quantities ξ vary along the chain smoothly such that $\xi_{l\pm 1}(x \pm a)$ can, to a good approximation, be written as $\xi(x) \pm a\partial_x \xi$. Employing the two-spin block partitioning we can use the equation of motion (3.5). Expanding now around $\xi_o(x)$, $\xi_e(x+a) = \xi_o(x) + a\partial_x \xi_o(x)$, and rescaling x as before, we obtain

$$\dot{\mathbf{N}} = \frac{1}{2}\partial_x \xi_o \left(\mathbf{N} \times \partial_x \mathbf{N} + \mathbf{N} \times \partial_{xx} \mathbf{N}\right) + \xi_o \mathbf{N} \times \partial_{xx} \mathbf{N} \ . \qquad (5.1)$$

"Sii

For example, for $\xi_o(x) = 1 + \Delta f(x)$ where $\Delta < 1$ is a parameter, the solution can be written in the form $\mathbf{N} = \mathbf{N}_F + \Delta \mathbf{N}_1$, where $\mathbf{N}_1$ is a small correction to the dominant well-known homogeneous ferromagnetic solution derived from (3.6).

B. *Strongly disordered chains*

Next assume that ξ_o is a stochastic random variable along the chain. If it fluctuates narrowly around a predominant ferro- or antiferromagnetic form it is possible to approach the problem by solving for the fluctuating part around the known respective exact solutions. E.g., suppose that $\xi_o = 1 + \zeta(x)$ with $\langle \zeta(x) \rangle = 0$ and $\langle \zeta(x)\zeta(x') \rangle = f(x - x')$, with the decay lengthscale, λ, of the correlation function f specified explicitly. If λ is finite we can partition the chain into sections of lengths $l < \lambda$, solve within each section by perturbing around the ferromagnetic and glue the solutions together by requiring that the boundary conditions match between neighbouring sections. (That's a project in itself that we haven't yet attacked properly. RB).

If, however, ξ fluctuates wildly between ± 1 with very short correlation length one needs to go back to the equation for $\mathbf{N}$, (3.5), and try to solve it from there. (Again, a project that has not been attempted yet. RB)

In any case, the important issue for strongly disordered chains is not as much finding the exact solutions for a given configuration of ξ, but rather inferring about the statistics of the solution from the statistical properties of the distribution of ξ along the chain. (I wonder if it is superfluous to write here the dynamic structure factor of the solution in its general form. It may be possible that by integration by parts one can use the equation of motion and possibly relate it to the statistics of ξ. RB).

6. Statistics for the homogeneous antiferromagnetic chain

Here we write down the dynamic structure factor for the multi-twist solutions found in [4]. (David, some of this is done, but I have some thoughts regarding this issue that I would like to discuss before continuing. RB)

7. Concluding remarks and discussion

We have done this and this and that.

REFERENCES

"mikes 1. see, e.g., the review by Mikeska and Steiner.

"bala 2. The works of Radha and collaborators.

"belpol 3. Belavin-Polyakov equation.

"bb 4. Blumenfeld and Balakrishnan; Blumenfeld and Saxena.