Stress transmission in disordered solid foams and cellular systems and a new state of solid matter

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Abstract

This paper develops a first-principles theory for stress transmission in planar cellular systems and solid foams. The theory is constructed on the cellular level and then upscaled to the macroscopic level. It is made possible by first developing a mapping from planar cellular networks to marginally rigid granular assemblies of equivalent geometry. General asymmetric networks are mapped onto assemblies of rough grains, while symmetric networks, where all edges meet at angles of $2\pi/3$, are mapped onto assemblies of smooth grains. The constitutive equation turns out to be sensitive to the local geometrical chiral disorder in the cellular structure via a new fabric tensor. A coarse-graining procedure is then proposed for upscaling the constitutive equation. An intriguing consequence of the analysis is that the stress in cellular systems under external loading can be determined in terms of the local geometrical structure alone, independent of stress-strain information. For assemblies of rigid grains marginal rigidity is a new state of solid matter and this may imply that cellular materials are also in this new state.

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I. Introduction

Planar-like cellular structures consist of partitions of space by irregular cells and are observed in many natural and man made systems. Their ubiquity in nature is often due to evolutional processes that have led to an optimization of mechanical or transport properties against weight. Much effort ha been invested in attempts to follow Mother Nature's wisdom and develop materials with biologically inspired structures. Essential to this is the development of a basic understanding of the relations between the structure and the macroscopic properties [1]. This is relevant to wide range of issues from design of high-perfrmance and smart materials to bio-artificial applications and the food insustry. A corner stone in this direction has been the work of Gibson, Ashby and collborators [2] who employed various approximations on ordered cellular structures to estimate the mechanical properties of cellular networks. While that work has significantly improved the theoretical basis for stress calculations, it has failed to provide a rigorous relation between the local cellular geometry in general disordered systems and the local stress field. There exists a growing body of knowledge on how to characterize the structural characteristics and understand the geometry of the cells in terms of the nonequilibrium dynamical processes that generate them. Yet, it has become evident that the lack of a first principle relation between the geometry and stress transmission stands in the way of progress on many fronts.

The conventional approach to relate constitutive structural data to macroscopic properties in general elastic solids is by an integration over microscopic variables [3]. This involves starting from a given constitutive stress-strain relation on a judiciously chosen microscopic scale and coarse-graining this relation self-consistently. This paradigmatic approach has been automatically adopted for cellular solids [4]But is it really necessary to go down this route?

In general terms, the problem is the following: Consider a cellular structure loaded through its boundaries by a set of forces, $\vec{g}$. Given the detailed geometrical structure of the network and the properties of the material that constitutes the network, the questions are: (i) how to describe the discrete field of forces transmitted by the cell walls as a continuous stress field? and (ii) what are the equations (the stress transmission equations) that govern this field? Formally, the problem can be posed as follows: In d dimensions it takes d^2 conditions to determine the $d \times d$ stress tensor σ. Force balance gives d conditions,

$$\nabla \hat{\sigma} = \vec{g} \tag{1}$$

and blanace of torque provides further $d(d-1)/2$ conditions,

$$\sigma_{ij} = \sigma_{ji} \tag{2}$$

These give $d(d+1)/2$ relations altogether and therefore additional $d(d-1)/2$ equations (one in $d=2$ and three in $d=3$) are required to solve for the stress components. In conventional elasticity theory this second set of equations is provided by first imposing compatibility conditions on the strain field and then relating the stress and the strain fields by an independent constitutive relation.

Equations (1) and (2) torque the stress to the external loading while the second set of equations relate the stress to the specific constitution of the medium. Differently stated, it is the constitutive relations that determine how the stress is affected by spatial variations of the medium properties. In cellular systems, these variations are governed by the local geometrical structure and therefore the structural characteristics must play a key role in determining the local stress. Yet, quantifying this relation from first principles has remained an open problem so far. Moreover, it has been unclear whether to obtain such a relation one needs to translate knowledge of the microstructural data into stress-strain relations and coarse-grain those or is there another.

A recent development in the context of two-dimensional granular systems has provided an unexpected key for progress on this problem. In a recent paper, Ball and Blumenfeld (BB) [5] have formulated a new theory for stress transmission in two dimensional marginally rigid granular systems [6] deriving a constitutive equation which torques the local stress to the local geometry. Marginal rigidity is a new state of solid matter where the microscopic force field, and therefore the macroscopic stress, can be determined without the use of any stress-strain data. BB have further shown that their theory predicts that the symmetric divergence-free stress field is exactly the Airy stress function [7], in perfect agreement with conventional elasticity theory. The derivation of Airy's stress function independently of stress-strain relations confirms their redundancy in those systems.

The aim of this paper is to derive a constitutive relation between the geometry of planar cellular structures and the stress field that develops in them. To this end, an exact mapping is constructed between cellular networks and granular assemblies at the marginal rigidity state. This mapping will make it possible to adapt the BB formalism and derive the desired relation, completing the stress transmission equations. However, the original theory of BB suffers from an inherent problem that the consitutive equation cannot be coarse-grained in a straightforward manner. This makes it difficult to apply the theory as is for obtaining a macroscopic equation for general systems. A new method is proposed here to overcome this difficulty. A significant consequence of the new theory is that the stress field in cellular systems is independent of the compliance of the material that comprises the cell walls. In the context of granular systems, this makes marginal rigidity a new state of solid matter and a discussion is presented on the relevance of this interpretation to cellular networks.

The paper is structured as follows: In section II a discussion is presented, leading to the conclusion that the stress field of an isostatic cellular network can be determined from its geometry alone, independent of any strain-stress information. Section III formulates a one-to-one mapping between general isostatic planar trivalent cellular networks and assemblies of rough grains with an equivalent geometry at a marginally rigid state. Section IV describes the derivation of the constitutive equation for planar systems. Section V tends to a difficulty in the upscalability that bedevils the BB theory and a method is proposed to resolve this problem. This method applies to both cellular and granular systems and completes the formalism into a fully macroscopic theory. Section VI presents a discussion of the results and some of their more intriguing implications.

II. The stress field can be determined independently of stress-strain relations

Consider a planar trivalent cellular network consisting of $N \gg 1$ cells [8]. The system is isostatic under a set of external forces acting on its boundaries. This loading induces tension or compression in the cell walls, balanced by forces developing along the walls. It also gives rise to torques on the vertices of the network, which must be balanced by forces that are normal to the walls. By Euler's theorem [9] a cell has on average six neighbors and is therefore surrounded by six cell walls and six vertices. Each a cell wall is shared by two cells and so there are altogether $3N$ walls. Furthermore, since each vertex in a trivalent network is the meeting point of exactly three walls and each wall connects two vertices then there are exactly $2N$ vertices. This enumeration neglects boundary effets, which introduce corrections of order $O(1/N)$.

Let us construct around every vertex a triangle whose corners are located at the midpoints of the walls that branch from it (see figure 1). Each of the 2N triangle is in contact with three neighbours and the triangles form a rigid frame which is capable of transmitting the same forces and stresses as the original cellular network. Each contact between neighbouring triangles transmits a force between them that is composed of a component along the cell wall (the tension or compression in the wall) and a component normal to the wall at that point (representing the normal force that contributes to balancing the torques on the vertices due to the external loading). Since there are 3N cell walls then this gives altogether $6N$ unknowns in the system.

How many equations are available to determine these unknowns? At mechanical equilibrium each triangle must obey two equations of force balance and one equation of torque balance. Therefore, the $2N$ vertices give $6N$ conditions, exactly enough to determine all the unknown forces. But the balance equations contain no information about the compliance of the network and in particuar it is independent of the properties of the material that comprises the cell walls. It depends only on the geometrical / structural details of the network and on the external loading.

Now, the macroscopic stress is just a coarse-grained description of the discrete forces on the cellular scale. Since the latter can be determined without reference to compliance then the macroscopic stress should be independent of this information as well. It is evident then that the second set of equations for the stress transmission must arise from the structural characteristics. Yet, a consitutive eqaution relating the local geometry to the stress in general systems has not been derived in the literature. Before continuing to the derivation of this relation it is important to note a significant ramification of the above argument: regardless of the precise form that the consitutive equation takes, none of the stress transmission equations involves information about the stress-strain response. It follows that stress transmission in cellular networks is governed by *physical laws that are different than in conventional solids.*

This basic realization appears to have been missed or neglected in the large body of literature on cellular materials. Most works in the field employ stress-strain relations, inherently linking the stress to the displacement that occurs as the system approaches mechanical equilibrium. The fundamental concept underlying this paper is that in cellular networks the stress can be decoupled from any deformation. Thus, after all displacement has stopped, the stress can be calculated from the isostatic configuration and the boundary conditions alone. Since stress transmission is the signature of a state of matter then it appears that cellular solids are not in the same state as conventional solids, an issue that will be discussed later on. The argument presented above can be extended to three dimensional cellular structures, but this will be shown elsewhere.

III. Mapping cellular networks onto granular assemblies

A planar cellular systems can be characterized by the positions of all its vertices, v, knowledge of the connectivity matrix, namely, which vertices are connected by cell walls, and by the curvature of the walls. The connectivity matrix is subject to the constraint that each vertex connects to exactly three neighbors. Thus defined, the cell walls tesselate the system into a set of cells, c. In addition to the geometry, one has to specify the external loading on the system, which is transmitted through the system via the cell walls. I shall first discuss networks where the angles between the walls meeting at each vertex are arbitrary, and then turn to analyse networks where all such angles are exactly $2\pi/3$.

The first step is to construct the dual network shown in figure 1: Around each cell connect the midpoints of the walls branching from it by straight lines. These lines form triangles around the vertices and polygons enclosed inside the cells. Note that the cell walls themselves need not form straight lines. Each side of such a triangle lies between a particular vertex v and a particular cell c and can be uniquely indexed by them. The sides are made into vectors, $\vec{r}_{cv}$, by assigning them directions so that they circulate the cells in an anticlockwise direction and consequently in a clockwise direction around the vertices (see figure 2).

The centroids of the cell polygons lie at the mean positions of the wall midpoints. It is also useful to define vectors extending from vertex v to the centroid of a neighbor cell c, $\vec{\rho}_{cv}$. The vectors $\vec{r}_{cv}$ and $\vec{\rho}_{cv}$ form two dual networks, each of which isostatically supporting the external loading.

We can now map the cellular network onto a virtual assembly of grains. The grains are centered on the vertices and make contacts at the points on the cell walls where the triangles touch. A contact between vertices v and v' transmits a force $\vec{f}_{vv'}$ that consists of a component along the wall, identified with the wall tension $T_{vv'}$, and a component $F_{vv'}$ normal to the wall, which assists in balancing the torques on vertices v and v' (see figure 3) [10]. The grains are regarded as infinitely rigid and of sufficient roughness to prevent slip at any of the contacts.

There are two notable differences between real cohesionless granular assemblies and those described above. The first concerns the direction of the contact forces. Real assemblies can only support compressive forces while tensile forces take them out of mechanical equilibrium. In contrast, cell walls can sustain either tensile or compressive forces. Correspondingly, the assemblies that cellular networks map onto, can also have either type of forces between the virtual grains [11]. Although the original BB formalism has been developed for real systems under compressive forces, it can accommodate tensile forces as well. The second difference is more fundamental and has far-reaching implications. The virtual assemblies are at the so-called marginal rigidity state [12], namely, at a state where all the intergranular forces can be determined from balance equations alone. The basic condition for this is that the average coordination number is three per grain (or six for perfectly smooth grains). However, nothing in the consolidation dynamics of real granular assemblies constrains them to end up with this average value. In fact, most granular piles in nature have a higher coordination number. In contrast, the above mapping of cellular systems generates assemblies of grains with exactly three force carrying contacts and, therefore, these are always at this state. The implications of this to the state of matter of solid foams and cellular materials are discussed in section VI.

IV. The missing constitutive equation

This section outlines the original BB theory in the context of cellular networks and derives the constitutive equation. It also highlights a difficulty of the original theory which concerns its upscalability and which will be resolved in section V.

A theory for stress transmission in any medium must provide the continuous missing equations to augment eqs. (1) and (2). To this end one first assigns each cell a force, $\vec{f}_c$. The contact forces $\vec{f}_{vv'}$ are then parameterized in terms of the cell forces as follows: Each contact point sits between exactly two cells, say l and m in figure 3. The relation between the contact forces and the vertex forces is given by

$$\vec{f}_{vv'} = \vec{f}_l - \vec{f}_m \tag{3}$$

The sign convention in eq. (3) is such that $\vec{f}_c$ is positive when the directed loop formed by the vectors $\vec{r}_{cv}$ around cell c is *toward* vertex v at the vv'-contact and vice versa for vertex v'. This parameterisation has several advantages: (i) The sum of the forces around every cell grain, $\sum_{v'} \vec{f}_{vv'}$, vanishes identically because each cell force contributes to this sum exactly twice, once with a positive and once with a negative sign. Thus, the forces around each vertex automatically balance. (ii) By construction, these forces satisfy Newton's third law of action and reaction, $\vec{f}_{v'v} = \vec{f}_m - \vec{f}_l = -\vec{f}_{vv'}$. (iii) There are only N vector forces to solve for now instead of the former $3N$, namely, two degrees of freedom per cell rather than two per cell wall. This corresponds to a reduction in the number of equations since the force balance constraints have been now satisfied. Indeed only the (one) torque balance equation for each of the $2N$ triangles remains unsatisfied, exactly accounting for the remaining variables to determine.

The next step is to smooth the discrete field of cell forces into a continuous field, $\mathcal{F}$. Regarding every cell force as located at the centroid of its cell, the value of the continuous force field at the vertex v is

$$\mathcal{F}_v = \frac{1}{3} \sum_c \vec{f}_c \tag{4}$$

where the sum is over the three cells surrounding v. The force field is made continuous by piecewise linear interpolation between the cell forces. Due to the linear interpolation, each cell force around vertex v can be presented in the form

$$\vec{f}_c = \mathcal{F}_v + \vec{\rho}_{cv} \cdot \vec{\nabla}\mathcal{F} \tag{5}$$

This expression is exact by construction and should not be confused with a first order truncation of a gradient expansion. In terms of the above variables, it can be shown that the traditional force moment, $\vec{R} \times \vec{f}$, on grain v can be written as

$$\hat{S}_v^{ij} = \sum_c \vec{r}_{cv}^{\,i} \vec{f}_c^{\,j} \tag{6}$$

where the sum run over the cells surrounding grain v. This, in turn, can be recast in the form

$$\hat{S}_v = \hat{C}_v \cdot \vec{\nabla}\mathcal{F} \tag{7}$$

where

$$\hat{C}_v = \sum_c \vec{r}_{cv}\vec{\rho}_{cv} = \left(A_v\hat{\epsilon} + \hat{P}_v\right) \tag{8}$$

and the superscript notation has been suppressed. $\hat{C}_v$ is a fabric tensor that characterizes the local geometry around vertex v and plays a crucial role in the following analysis. A_v is the area around the vertex v shown shaded in figure 2, and $\hat{\epsilon} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the unit antisymetric tensor. The entire system can be conveniently covered by these areas, $A_{sys} = \sum_v A_v$. The first and the second terms on the right hand side of eq. (8) are, respectively, the antisymmetric, $(\hat{C}_v - \hat{C}_v^T)/2$, and symmetric, $(\hat{C}_v + \hat{C}_v^T)/2$, parts of fabric tensor. Rewriting the vectors $\vec{r}_{cv}$ and $\vec{\rho}_{cv}$ in expression (8) in terms of the vectors $\vec{s}_{cv}$ and $\vec{t}_{cv}$ shown in figure 2, it is straightforward to show that the symmetric part of $\hat{C}_v$ can be represented in the following form

$$\hat{P}_v = \frac{1}{2}\sum_c \left(\vec{s}_{cv}\vec{s}_{cv} - \vec{t}_{cv}\vec{t}_{cv}\right) \tag{9}$$

The departure of $\hat{P}_v$ from zero around vertex v is a direct measure of how much the v triangle is rotated relative to its immediate environment. In other words, this tensor characterizes the local *chiral* disorder.

Applying now the yet unused requirement that the triangles around the vertices obey torque balance, BB have arrived at two conditions: One reduces to the requirement that the *mean* stress field is symmetric, which fits the expectation from a global consideration of torque balance. The second condition amounts to requiring that the antisymmetric part of the tensor $\hat{P}\hat{\epsilon}^T\hat{\sigma}$ vanishes. Defining a rotated version of $\hat{P}$, $\hat{Q} = \hat{\epsilon}\hat{P}\hat{\epsilon}^T$, this condition can be written as

$$\hat{Q} : \hat{\sigma} = 0 \tag{10}$$

This is the missing constitutive equation! It couples the local stress to the local fluctuations in the geometry of the vertices via a fabric tensor and completes the set of equations needed to determine the stress field. Jusit like the tensor $\hat{P}$, the fabric tensor $\hat{Q}$ characterizes the local chirality. Thus, eq. (10) couples the stress field to this particular aspect of the geometry.

In granular assemblies the formulation of the theory has been based on the assumption that the grains are sufficiently rough so that there is no slip at any contact point. BB [5] have shown, however, that it can also be applied to assemblies of perfectly smooth grains that transmit only normal forces. This has been achieved by first mapping the $\mathcal{N}$-grain 'smooth' system into a system of $3\mathcal{N}/2$ rough grains, applying the theory, and then tending the extra grains to zero.

V. Upscaling the constitutive equation

In principle, the above should conclude the derivation of the equations of stress transmission. Unfortunately, the problem is not resolved entirely. The reason is that, for the equations to be useful on a scale relevant to engineering, one has to be able to identify nontrivial coarse-grained values for the components of the fabric tensor $\hat{Q}$. This turns out to be quite tricky. The difficulty lies in the fact that the area average of $\hat{P}$ tends to zero. To see this choose a patch Γ inside the system, of area A_Γ and boundary $\partial\Gamma$, which contains $\mathcal{N}$ vertices. The average of $\hat{P}$ over this patch is

$$\hat{P}_\Gamma \equiv \langle \hat{P} \rangle_\Gamma = \frac{1}{A_\Gamma} \sum_{v \in \Gamma} \hat{P}_v = \frac{1}{A_\Gamma} \sum_{c,v \in \Gamma} \left(\vec{s}_{cv} \vec{s}_{cv} - \vec{t}_{cv} \vec{t}_{cv} \right) \tag{11}$$

It can be verified by inspection that neighboring grains v' and v, sharing cell c, obey $\vec{s}_{cv} = -\vec{t}_{c'v}$. Therefore, the right hand side of (11) vanishes identically under summation over every cell fully contained in Γ. It follows that the average vanishes up to terms arising from $\partial\Gamma$ that do not cancel. The contribution of these then is of order $1/\sqrt{\mathcal{N}}$ and tends to zero for macroscopically large systems.

This introduces a complication when we come to apply the formalism to the macroscale. The usual procedure is to choose a particular lengthscale, L, and divide the entire system into areas of linear size L, each of which being regarded a basic unit. Each such unit is then assigned a mean characteristic constitutive property, in our case this would be the values of the components of the $\hat{P}$ tensor. Using this in the constitutive equation, one derives the upscaled stress field. The vanishing of the mean of $\hat{P}$ within the unit, however, undermines this approach making the coarse-graining far from straightforward. To address this issue we use an observation of BB that this problem is successfully resolved when granular assemblies have a so called staggered order. In the context of planar cellular netwroks, staggered order arises when all the cells have even number of edges. In such networks, it is possible to label the vertices + and -, such that each vertex has neighbors of only opposite sign. The entire network can then be covered by pairs of +/- units, each of which possessing an area, $A_{pair} = A_v^+ + A_v^-$. The mean pair stress is their average force moment normalized by the area, $\hat{\sigma}_m = (S^+ A_v^+ + S^- A_v^-)/A_{pair}$, and we also define $\hat{\sigma}_d = (S^+ A_v^+ - S^- A_v^-)/A_{pair}$. Both these stresses must be symmetric to satisfy torque balance and so they give [5] eq. (2) and a modified version of eq. (10) [13]

$$\hat{Q}^+ : \hat{\sigma} = 0 \tag{12}$$

Unlike the average of $\hat{Q}$, that of $\hat{Q}^+$ does *not* vanish identically and can be upscaled. This alleviates the difficulty for networks with a staggered order.

But we are still faced with a problem in general structures where the staggered order is broken at places. On inspection, we can see that the problem stems from the occurrence of cells with odd number of edges

which cannot accommodate perfectly alternating signs of vertices around them. Let us then consider such an odd-edged cell (OEC) embedded in a region of the network that consists only of even-edged cells. Starting from an arbitrary vertex, label the vertices alternatingly + and - along the perimeter of the OEC, say in the clockwise direction. We are left with two 'frustrated' neighbors having the same sign, say -. Now apply the same procedure to the first shell of vertices surrounding the OEC, starting from a neighbor of a frustrated vertex as shown in figure 4. It can be readily verified that this first shell must also contain a frustrated pair of vertices that lies adjacent to the previous one. Repeating this process on the next shell and continuing it outwards results in a line of frustrated pairs of vertices emanating from the OEC and extending indefinitely. The only way that this line can be capped is when it encounters another OEC along its path. This suggests a well defined procedure to identify the frustrated pairs as follows. First mark all the OECs in the network and partition them into close-by pairs. Between each such pair draw a line of frustrated vertex pairs according to the above description. Except along those frustrated lines of -/- pairs, the system of vertices is fully ordered, i.e., is completely covered by pairs of +/- vertices.

For the purpose of upscaling the consitutive equation we consider each pair of nearest neighbor frustrated vertices and the cell wall connecting them as one 'fused' vertex. This makes no difference to the physics because the corresponding triangles are rigidly linked anyway under the conditions of the problem. The fusion eliminates a contact point and, in a way, it is natural from the mathematical point of view. This can be seen by considering how the geometric tensor $\hat{C}_-$ renormalizes under this operation, $\hat{C}_- = \hat{C}_1 + \hat{C}_2$. First, the area of the fused pair, being the prefactor of the antisymmetric part of $\hat{C}_-$, is the sum of the bordering areas of the individual constituents. Second, the renormalized symmetric part of the fused pair is $\hat{P}_- = \hat{P}_1 + \hat{P}_2$. Using the geometrical interpretation of $\hat{P}$ (eq. (11)), it can be observed that the vectors $\vec{s}$ and $\vec{t}$ connecting the eliminated contact point to the centres of the neighboring cell loops, cancel out, naturally eliminating any effect of that point. Therefore, this practical idea of fusion is facilitated in a straightforward manner by the mathematics. The fusion of pairs along each frustration line in the system eliminates the local 'frustration' and the network acquires a staggered order everywhere. Such systems are upscalable, as described above. The procedure proposed here is well defined and is somewhat analogous to capping of branch-cut lines between pairs of logarithmic singularities in the complex plane. It should be noted that the algorithm of choosing the lines is not unique due to the seemingly arbitrary choice of the OEC pairs. To make it unique one can adopt the conditions (i) that the frustration lines do not intersect and (ii) that the choice of pairing minimizes the total length of these line. Using this procedure, the BB equations can now be coarse-grained to any cellular or granular system and the above formalism becomes a usable macroscopic theory for stress transmission in solid foams.

Symmetric trivalent networks

In symmetric trivalent networks the cell walls meet at each vertex at angles of $2\pi/3$. This structure abounds in nature, particularly in dry foams and a variety of froths. In contrast to the general case, here force balance at every vertex dictates that all walls are under the same tension or compression. Since only the normal force on each cell wall is unknown the number of degrees of freedom is reduced from $6N$ to $3N$. This appears to invalidate the foregoing analysis. Nevertheless, it is possible to use the formalism developed above as follows. First, we formulate a new mapping with the normal forces translating into contact forces between grains. The geometry of the new construction is similar to the previous one except that the virtual grains are now centered on the cells rather than on the vertices. Each grain now contacts six neighbors and the contact points are located again at the cell wall midpoints. The virtual grains are perfectly smooth and can only transmit forces normal to the surface at the contact. Therefore there is only one degree of freedom at each contact, giving altogether $3N$ unknowns. Each of the virtual grains satisfies two force, and one torque, balance equations, giving the $3N$ conditions needed to determine all the unknowns. So, the intercellular forces can again be determined without any information about the compliance of the system and this granular assembly is also at a marginally rigid state.

To adapt the BB formalism, the system of smooth grains needs to be further mapped onto an equivalent assembly of rough grains. To this end we follow the procedure outlined in reference [5]. We insert at each contact a vanishingly small ball bearing between the grains (see figure 5). We now regard the grains and the ball bearings as having an infinitely high friction coefficient. This does not alter the intergranular forces because the ball bearings anyway transmit only forces that are normal to the surfaces of the grains that they touch. Therefore, the original distribution of intergranular forces is recovered in the limit when the radii of the ball bearings tend to zero. After inserting a ball bearing b between two grains centered on c and c' the contact forces are $\vec{f}_{cb} = \vec{f}_{cc'}$ (see figure 5). In the original cellular system the average coordination number per cell six (it would be exactly six if none of the walls were curved)and so there are on average three contacts per grain. This gives three additional ball bearings for each original cell grain, each of which with coordination number two. The new system now consists of $4N$ grains altogether and the new mean coordination number is $(6N + 2\text{x}3N)/4N = 3$. This is exactly the correct number to determine the intergranular forces in a general assembly of rough grains. Thus, all assemblies of smooth grains can be mapped into rough ones with equivalent geometries for which the BB theory applies. Therefore, symmetric cellular networks can be described by the above formalism.

VI. Conclusion, discussion and future directions

In this paper I have shown that all planar trivalent cellular systems can be mapped onto granular assemblies of infinitely rigid grains at the marginal rigidity state. Networks, with cell walls meeting at arbitrary angles map onto assemblies of rough grains while symmetric networks map onto ones of perfectly smooth grains. The mapping constructed here enables the application of the Ball-Blumenfeld formalism for deriving the constitutive equation for stress transmission in solid foams. This equation couples the local stress to the local chiral disorder and replaces the traditional stress-strain relations. I have further addressed the problem of the coarse-graining of the constitutive equation, thus extending the BB formalism and making it applicable for macroscopic systems.

One striking consequence of the analysis presented here is that isostatic planar cellular systems under external loading are *always in a state of marginal rigidity.* This new state of solid matter has so far only been discussed in the context of granular assemblies [14] and this is the first time it is identified in cellular systems. Furthermore, while in granular assemblies this stae can only be approached under careful consolidation processes [14], trivalent cellular systems are naturally in this state.

The analysis presented in this paper has many fundamental and practical implications, some of which are the following:

Fundamental issues:

(i) It has been shown by BB that the new equations can recover the fundamental result that the symmetric divergence free stress field is the Airy stress function, a result usually obtained within conventional elasticity theory. Beyond supporting the new approach, this points to the redundancy of the stress-strain relations in marginally rigid systems.

(ii) The new constitutive equation can be used on scales that range from a couple of cells to macroscopic sizes due to the coarse-graining procedure proposed here. This suggests two potentially fruitful directions: First, a test of the formalism by its application to systems where the stress transmission is well understood (e.g., ordered cellular networks such as the honeycombe or brick-like). Second, investigation of disordered cellular structures where there currently exist only approximate effective calculations.

(iii) Intertestingly, in granular assemblies marginal rigidity lies between the fluid and the traditional solid states. This observation appears to have no analogue in isostatic cellular solids. It may, however, be relevant in the context of the rheology of cellular materials close to the jamming point, as well as for the dynamics of cell formation. A connection between the constitutive equation and rheology in granular assemblies has been recently suggested [15] and it would be interesting to try to extend it to the flow of foams, an issue which is of immediate relevance to many technological applications.

(iv) The basic principle of force determination on the cellular level without resorting to information about

the compliance of the cell walls does not appear to depend on the two-dimensional nature of the system and may well be extended to three-dimensions. In particular, it is conjectured here that it is possible to extend this principle to solid foams where each vertex is the meeting point of four edges and six walls, a structure which is ubiquitous in nature. The basis for this conjecture is the observation that in these structure it is possible to calculate, in principle, the force distribution in the cell walls from balance equations alone. This suggests that a larger class of solids are in the new state of marginal rigidity and therefore that they can also be addressed outside elasticity theory.

(v) The theory developed here goes outside the existing paradigm. Although the discussion focused on the difference between it and elasticity theory, one should recall that *all* conventional theories of stress transmission are based on a constitutive relation that relies on some sort of deformation. Elasticity employs strain while elastoplastic and viscoelastic theories are based on strain rate. Thus, the conventional paradigm always relates the stress to a virtual deformation, while the present theory does not. This then begs the question what is its exact relation to conventional practice. Elasticity theory, in particular, is very successful when the slightest strain is allowed. This leads to the following interesting dillema: consider a solid foam of the type discussed here, which is allowed an infinitesimally small strain before it settles in its isostatic state. Elasticity theory should be able to calculate the stress just before the isostatic state is reached while the new theory should be able to do so after. It stands to reason that the stress field cannot jump discontinuously once the infinitesimal strain has been applied and therefore both theories must yield in this limit the same answer. This issue is being currently under investigation and preliminary results suggest that requiring this condition leads to explicit constraints that the strain-stress relations of elasticity theory must satisfy.

Practical issues:

(i) First and foremost, this theory will clearly make it possible to model disordered planar cellular materials to predict stress transmission, extending the current capabilities.

(ii) The theory should make it straightforward to predict 'hot spots' of stress concentration in such systems and make useful connections between the occurrence of hot spots and the local structure. This knowledge is expected to both improve prediction of failure in advanced materials and set new goals for material processing methods to manipulate structural characteristics in order to avoid such vulnerability.

(iii) It may be possible to exploit the formal relation between granular assemblies and cellular systems to transfer results between the two fields, as well as carry out experiment on one class of systems in order to learn about the other. For example, recall that cellular systems are naturally at the marginal rigidity state while granular assemblies can only approach it under careful preparation. Therefore, it is much more convenient to use the former to experimentally test ideas pertaining to the latter. The information extracted

from such experiments should prove to be clearer than direct experiments on granular assemblies, which are usually prone to the large fluctuations inherent to the dynamics of consolidation.

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5. R. C. Ball and R. Blumenfeld, Phys. Rev. Lett., 88 115505 (2002).
6. Although the formalism has been developed in the context of rough and infinitely rigid grains it appears to apply to systems of finite rigidity as well. See, R. Blumenfeld, in preparation.
7. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, (P. Noordhoff, Groningen, Netherlands 1963).
8. The larger the number of grains the more negligible are the effects of the boundaries.
9. This is a direct reult of Euler's theorem, proven circa 1750, which relates the number of cells, faces and edges.
10. Although the normal forces are distributed along the cell walls, it is possible to show that these can be uniquely represented by the localised force $F_{vv'}$ acting at the contact point.
11. It should be noted that depending on the boundary loading, the stresses developing in a network would be either all compressive, or all tensile. For a proof of this see J. C. Maxwell, Phil. Mag., Ser 4, **27**, 250 (1864); J. C. Maxwell, Trans. Roy. Soc. Edinb., **26**, 1 (1869).
12. R. C. Ball, in *Structures and Dynamics of Materials in the Mesoscopic Domain*, eds. M. Lal, R. A. Mashelkar, B. D. Kulkarni, and V. M. Naik (Imperial College Press, London 1999); R. Blumenfeld, S. F. Edwards, and R. C. Ball, Nature, submitted.
13. To arrive at this equation, define $\bar{\hat{Q}} = (\hat{Q}^+ + \hat{Q}^-)/2$ and $\delta\hat{Q} = (\hat{Q}^+ - \hat{Q}^-)/2$. The torque balance eqution for the triangles riding the + and - vertices can then be manipulated to give $\delta\hat{Q} : \hat{\sigma} = 0$, where now $\hat{\sigma}$ is the mean stress over both triangles. Summing over the entire network we have $\sum \hat{Q} = \sum(\hat{Q}^+ + \hat{Q}^-) = 0$ and therefore $\sum \hat{Q}^+ = -\sum \hat{Q}^- = \sum \delta\hat{Q}$. This sum does not vanish identically, meaning that an average of $\delta\hat{Q}$ would be finite for disordered geometries and would give a macroscopic finite value..
14. R. Blumenfeld, S. F. Edwards, and R. C. Ball, Nature (2002), submitted.
15. R. C. Ball and R. Blumenfeld, in preparation.