The stress field in granular systems: Loop forces and potential formulation

Robin C. Ball[1] and Raphael Blumenfeld[1,2]
[1] *Dep. of Physics, University of Warwick, CV4 7AL Coventry, UK.*
[2] *Polymers and Colloids, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, UK*

We show that the transmission of stress through a marginally stable granular pile in two dimensions can be exactly formulated in terms of a vector field of loop forces, and thence in terms of a single scalar potential. The loop force formulation leads to a local constitutive equation coupling the stress tensor to fluctuations in the local geometry. If the local geometry of the pile has long range order of antiferromagnetic character, a mean field theory follows for the propagation of stress. Known exact lattice solutions for rough grains fall into this class, and we show that smooth grains can be mapped onto this. The disordered case has some spin-glass character, and may have no mean field description at all.

64.60.Ak, 05.10.c 61.90.+d

Significance of the problem, previous work, reference to the effect of the friction and the relation to the number of contact points between grains [1]

We discuss here two dimensional systems of perfect-friction grains that cannot slide against each other. The application of the results to frictionless systems will be shown below. For simplicity we focus on the equations governing a region transmitting stress but not otherwise directly loaded. The system under consideration consists of N grains [2] with intergranular compression forces transmitted via M contact points (CPs). At mechanical equilibrium both force and torque moment should balance on each grain, as well as the force at each CP. In two dimensions these constraints of mechanical equilibrium cannot be achieved by adjustment of the forces for $2M < 3N$, whilst for $2M > 3N$ the intergranular forces are underdetermined unless intrinsic mechanical properties of the grains are invoked. Our discussion focusses on the intriguing marginal case, corresponding to a mean coordination number $\bar{z} = 3$, where mechanical balance alone determines the intergranular forces and hence the transmission of stress through the system. This has been identified as a paradigm problem of theoretical granular mechanics [3] .
The macroscopic analogues of balancing force and torque are that the stress tensor σ obey

$$\nabla\hat{\sigma} = 0 \quad , \quad \mathcal{A}(\hat{\sigma}) = 0, \tag{1}$$

where $\mathcal{A}(\hat{\sigma}) = \frac{1}{2}(\hat{\sigma} - \hat{\sigma}^T)$ is the antisymmetric part of $\hat{\sigma}$. In two dimensions this system of equations is incomplete without one further *constitutive relation.* The central mystery we are seeking to unravel is the nature of this relation when it also arises from force balance at the granular level.
The key idea of this paper is to focus attention on local loops around contact points and corresponding loop forces. Each local void, which we will label by an index l, is enclosed by a loop of grains and contact points; each contact point separates two such voids and so is party to two such loops. Around each loop we take a *loop force* $\vec{f}_l$ to circulate in the anticlockwise direction, that is $+\vec{f}_l$ is contributed to each intergranular force around the loop with (notionally) positive sign in the anticlockwise direction, and vice versa for the reaction forces. The resulting total force across a contact point is then a difference between two contributing loop forces. The loop forces have two key features, the first being that they parameterise the intergranular forces in a manner that automatically satisfies balance of force at each CP and on each grain. In terms of them we can write the stress tensor on each grain as

$$\hat{S}_g = \sum_l \vec{r}_{lg}\vec{f}_l \tag{2}$$

where $\vec{r}_{lg}$ is the vector connecting (anticlockwise) the two contact points shared by loop l and grain g and A_g is the grain area defined in fig. 1 [4]. The vectors $\sum_g \vec{r}_{lg} = 0$ form an anticlockwise loop round the contact points of loop l and likewise $\sum_l \vec{r}_{lg} = 0$ forms a clockwise loop round the contact points of grain g and play a central geometrical role in our discussion.
The second key feature of the loop forces is that, being defined on the loops rather than the grains, they are a comparatively coarse-grained quantity. This follows because for $\bar{z} = 3$ one can

readily show that the number of loops is given by $L = N/2 = M/3$. The hidden nature of the constitutive equation can now readily be appreciated, because the only remaining constraints that the loop forces have to obey is balance of torque around each grain, that is

$$\sum_l \vec{r}_{lg} \times \vec{f}_l = 0 \tag{3}$$

which is one equation per grain and hence *two* equations per loop.

In essence the two torque equations per loop give us two macroscopic conditions, $\mathcal{A}(\hat{\sigma}) = 0$ *plus* a constitutive relation. To achieve this separation explicitly we postulate a smooth interpolation of the loop forces to a function of continuous position, $\vec{f}(\vec{r})$ with each loop and grain having a nominal centre $\vec{R}_l$ and $\vec{R}_g$, respectively. Then the force moment on grain g is, to a first order Taylor approximation,

$$\hat{S}_g = \hat{C}_g \cdot \vec{\nabla} \vec{f} \tag{4}$$

where

$$\hat{C}_g = \sum_l \vec{r}_{lg} \vec{R}_{lg} \quad \text{and} \quad \vec{R}_{lg} = \vec{R}_g - \vec{R}_l. \tag{5}$$

The tensors $\hat{C}_g$ characterise the grain local geometry and are readily shown to have the following useful properties. First

$$\hat{C}_g = A_g \hat{R} + \hat{P}_g \tag{6}$$

where $\hat{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the unit antisymetric tensor (also corresponding to $\frac{\pi}{2}$-rotation in the plane) and A_g is the area of the grain defined in fig. 2 **??**. The symmetric part, $\hat{P}_g$, reflects the departure of the local geometry from isotropy and its average tends to zero already on a scale of few grains. Consider then a region of area A whose boundary is defined by the CPs of the outermost grains. Averaging over $\hat{S}_g$ in the region using eq. (4) gives

$$\hat{\sigma} = \frac{1}{A} \sum_g \hat{C}_g \cdot \vec{\nabla} \vec{f} = \langle \hat{C}_g \cdot \vec{\nabla} \vec{f} \rangle \ . \tag{7}$$

Carrying out the same summation using eq. (2) we observe that the sum over loops inside the region cancels out and the only contribution comes from the boundary vectors, $\vec{r}_b$, between boundary CPs,

$$\hat{\sigma} = \frac{1}{A} \sum_{\text{boundary}} \vec{r}_b \vec{f}_b \ , \tag{8}$$

where $\vec{f}_b$ is the external loading. This contour sum can be converted using Stokes theorem into

$$\hat{\sigma} = \frac{1}{A} \int ds \hat{R} \cdot \vec{\nabla} \vec{f} = \hat{R} \cdot \langle \vec{\nabla} \vec{f} \rangle \ . \tag{9}$$

Since $\langle \hat{P}_g \rangle = 0$ we identify $\langle \hat{C}_g \rangle = A\hat{R}$ and, on comparing to eq. (7), we obtain

$$\langle \hat{C}_g \cdot \vec{\nabla} \vec{f} \rangle = \langle \hat{C}_g \rangle \cdot \langle \vec{\nabla} \vec{f} \rangle \ . \tag{10}$$

This result is of central interest because it yields the effective response of a macroscopic granular region to a force field in terms of an effective macroscopic characteristic property $\langle \hat{C}_g \rangle$. It is equivalent to results in contexts such as disordered dielectrics and continuum elasticity that are derived by combining a field equation with either a constitutive relation or an energy functional.

Thus the continuum stress tensor is $\hat{\sigma} = \sum_g \frac{1}{A_g} \hat{S}_g = \mathcal{D} \vec{f}$ where $\mathcal{D} = \hat{R} \cdot \vec{\nabla}$ is exactly the curl operator in two dimensions. The condition that the stress tensor on each grain be symmetric then leads, over and above the mean field condition that $\hat{\sigma}$ be symmetric, to the requirement

$$\mathrm{Tr}(\hat{Q} \cdot \hat{\sigma}) = 0, \tag{11}$$

where $\hat{Q} = \hat{R} \cdot \hat{P} \cdot \hat{R}^T$ is a rotated verion of the symmetric part of the local $\hat{C}_g$ tensor.

Within the approximation of the mean field treatment relation (11) is the constitutive equation which provides the missing link between the stress and the local geometry. It has the striking feature that the coefficients $\hat{Q}$ are spatially fluctuating quantities that locally add to zero, so that any simple attempt to identify a non-vanishing mean field value $\langle \hat{Q} \rangle$ leads back to Eq. (1) and therefore yields no new constitutive information. In this sense the problem is analogous to spin glasses, in our case it being the symmetric part of the stress tensor that is subject to spatial couplings of random sign. The spin analogy is useful in that it allows to identify cases with exact solutions as will be discussed below.

We now introduce an alternative formulation of the problem where the loop forces are defined in terms of a potential field. Since each loop is associated with a two-dimensional force the number of degrees of freedom that the potentials should provide is twice the number of loops, which (up to relative correction of $1/\sqrt{N}$ due to the boundary) is exactly the number of grains in the system. Thus, we require that the potential field provide one degree of freedom per grain leading to scalar grain potentials, ψ_g. The forces are derived from the potentials via

$$\vec{f}_l = \frac{-1}{A_l} \sum_g \vec{r}_{lg} \psi_g \ , \qquad (12)$$

and the potentials are determined from eq. (3). As for the force field, we assume that the ψ-field can be continued throughout the system and, expanding around the loop centres, we obtain

$$\vec{f}_l = \frac{1}{A_l} \hat{C}_l \cdot \vec{\nabla} \psi \ . \qquad (13)$$

The geometrical tensor $\hat{C}_l = \sum_g \vec{r}_{lg} \vec{R}_{lg}$ is the loop analogue of $\hat{C}_g$ and it consists of four independent degrees of freedom: three required for the symmetric part and one for the antisymmetric. As in the case of $\hat{C}_g$, the latter gives exactly the area of the loop A_l. $\hat{C}_l$ depends on the surrounding grain centres and since there are two grains per loop there are four degrees of freedom that can be chosen judiciously so that the symmetric part vanish identically, namely, $\hat{C}_l = A_l \hat{R}$. Expressing the local stress in terms of the potentials we therefore have

$$\hat{S}_g = \hat{C}_g \hat{R}^T \mathcal{D}_g \mathcal{D}_l \psi \qquad (14)$$

and the continuum stress becomes

$$\hat{\sigma} = \mathcal{D}_g \mathcal{D}_l \psi \ . \qquad (15)$$

It follows that the continuous version of ψ is exactly the Airy stress function.

To illustrate the utility of the above results we consider a periodic lattice (fig. 3) with a generally anisotropic unit cell comprising of two grains labeled + and -. Using eq. (4) for each grain separately, we write down the mean and deviatory cell stresses, for which we find

$$\hat{\sigma} = \left(\hat{S}_+ + \hat{S}_-\right) = \left(A\hat{R} + \hat{P}_+ + \hat{P}_-\right) \cdot \vec{\nabla} \vec{f} \qquad (16)$$

and

$$\hat{\delta} = \left(\hat{S}_+ - \hat{S}_-\right) = \left(\delta A\hat{R} + \hat{P}_+ - \hat{P}_-\right) \cdot \vec{\nabla} \vec{f} \ , \qquad (17)$$

where A is the unit cell area and $\delta A = A_+ - A_-$ is the difference between the areas occupied by the two grains. For the periodic lattice $\hat{P}_+ + \hat{P}_- \equiv 0$ and

$$\hat{P}_+ - \hat{P}_- = \frac{R}{2} \begin{pmatrix} \sqrt{3}(b_x - c_x) & a_x \\ \sqrt{3}(b_y - c_y) & a_y \end{pmatrix} \qquad (18)$$

which yields the following exact constitutive relation

$$\mathrm{Tr}\left[\hat{R}(\hat{P}_+ - \hat{P}_-)\hat{R}^T \hat{\sigma}\right] = 0 \ . \qquad (19)$$

The reason why the deformed honeycombe lattice is easy to solve is that it has an 'antiferromagnetic' order, namely, we can label its grains + or - such that each grain is surrounded only by opposite sign neighbours. This is the case in general for systems where each loop has an even number of edges. Regarding pairs of grains as basic units, we define the mean and deviatory cell stresses as in eqs. (16) and (17). Expressing $\hat{\delta}$ in terms of $\hat{\sigma}$ leads to

$$\mathrm{Tr}\left[\hat{R}\left(\delta A\hat{R} + \hat{P}_+ - \hat{P}_-\right)\left(\hat{R} + \hat{P}_+ + \hat{P}_-\right)^{-1} \hat{\sigma}\right] = 0 \ , \qquad (20)$$

which is the general constitutive relation for such systems.

Generally disordered systems can possess a predominant antiferromagnetic order but with defects where same-sign grains are forced to neighbor. This is not unlike local frustration in antiferromagnetic Ising systems, a similarity that suggests a possible use of methods from spin glasses to this problem.

The idea of a staggered order parameter can be further exploited to formulate a mapping from perfectly rough to perfectly smooth systems. Consider a system of perfectly rough grains where at each CP we insert a vanishingly small rough sphere between the two grains. The balance of forces $f_{\alpha\beta} = -f_{\beta\alpha}$ is maintained by

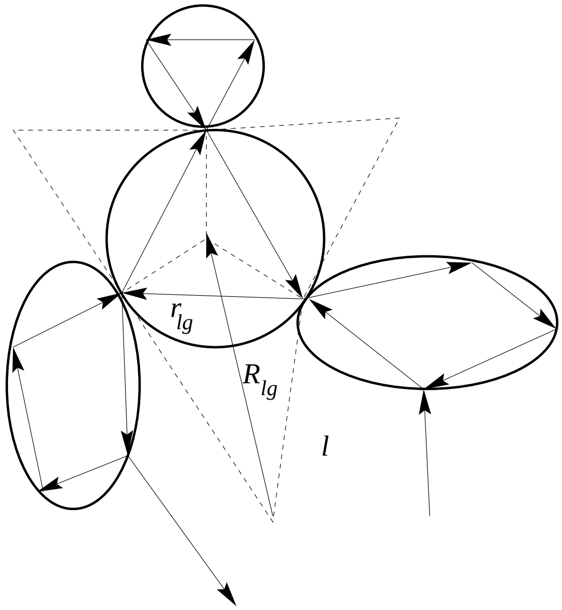

FIG. 1. Definition of the CP vectors and the grain area A_g.

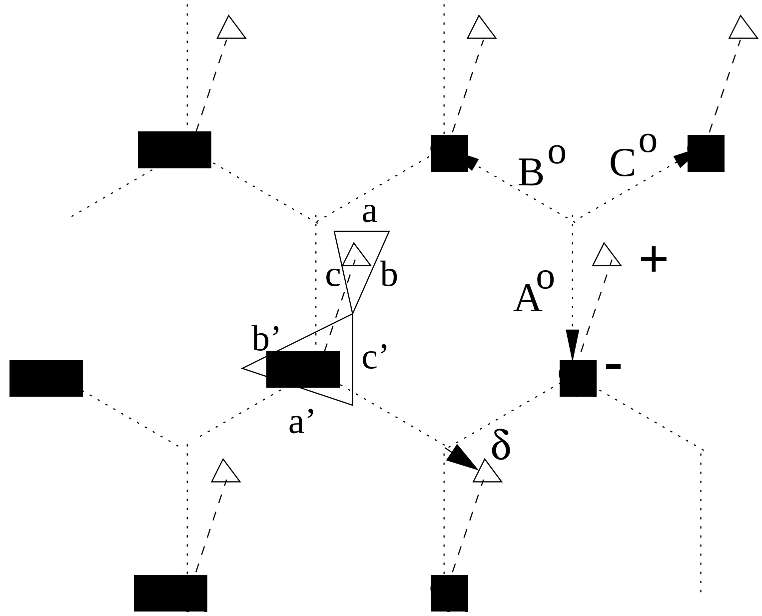

FIG. 2. Part of an anisotropic periodic lattice.

letting the forces act at antipodes and therefore the torque moments on these spheres vanish. If the rough system could be solved then $\bar{z} = 3$. The introduction of small spheres doubles the number of CPs and there are now 6 CPs per original grain, which is exactly the right number for a smooth system [1] to determine the stress uniquely. The smooth system is then obtained in the limit of the radius of the small spheres tending to zero. It follows that all the above results can be extended to systems of smooth particles.

---

[1] Previous work.
[2] A grain is defined as a rigid particle of finite volume and arbitrary shape.
[3] See eg, ...
[4] The definition of A_g as a grain area is useful because the entire system can be uniquely mapped in terms of these areas and $A_{system} = \sum_g A_g$.
[5] The centroid of a grain is defined as the point where $\sum_{g'} \vec{R}_{gg'} = 0$.
[6] Ball, Grinev, Edwards.

Comment A: Note that we are left with an arbitrary choice of loop centres $\vec{R}_l$, which play a role in determining $\hat{C}_g$. For the time being it is useful to fix these centres as the centroids of the loops so as to render $\nabla \vec{f}$ a well defined quantity. (Note further that once we define the forces in terms of potentials we need to discuss the geometrical tensors $\hat{C}_l$. Now, the arbitrary choice of both $\vec{R}_l$ and $\vec{R}_g$ needs to be considered. Since there are two grains per loop, the freedom in the choice of $\vec{R}_g$ translates into four degrees of freedom per $\hat{C}_l$. The antisymmetric part of $\hat{C}_l$ is the area A_l and therefore by adjusting three of the remaining degrees of freedom we can, barring a nasty surprise, make the symmetric part of $\hat{C}_l$ vanish identically, leaving one arbitrary degree of freedom per loop. Similarly, the freedom in the choice of $\vec{R}_l$ translates into one degree of freedom per $\hat{C}_g$. This leaves us altogether with 3 degrees of freedom per loop. Whatever that implies!)

Comment B:

Note that in (7) $\vec{\nabla} \vec{f}$ is a continuation of its value on the boundary of the region while in (9) it is a local quantity. Nevertheless, the two definitions are expected to converge in the continuum limit.