Stresses in disordered cellular solids: Explicit solutions for planar foams

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A new understanding is reported of stress transmission in disordered open-cell cellular solids, based on the observation that the skeletons of such structures are in an isostatic state. This obviates the conventional equations of elasticity theory and a new theory is proposed instead - isostaticity theory. The field equations of the new theory are given explicitly for two-dimensional systems, wherein the global force and torque balance equations are closed by stress-structure constitutive relations. This makes it possible to determine stresses independently of stress-strain data, leading to stress fields that differ significantly from those given by elasticity theory. Explicit solutions of the field equations are derived and discussed. The solutions are nonuniform and correspond to stress chains.

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I. Introduction

Disordered open-cell cellular structures (OCCS) abound in natural and biological materials. Examples include sponges, wood, cork, bone and porous rocks. These structures consist of frameworks of struts that make polygons in two-dimensions and polyhedra in three. Struts connect at junctions termed vertices and the number of struts converging at a vertex is known as the vertex valency, coordination number, or functionality, depending on the scientific discipline.

Cellular inanimate structures are often the consequence of basic physical mechanisms that govern the dynamics of the structural formation processes. In plants and other organic systems cellular structures are nature's way of optimising specific functionalities, often against economy in raw materials. There are only very few principles underlying the formation dynamics of bubbles and foams and therefore most common in nature and man-made materials are structures of valency three in two dimensions (trivalent) and of four in three dimensions (quadrivalent).

Inspired by Mother Nature, OCCS are rapidly becoming an important class of materials for a range of technological applications. One of the main advantages of cellular solids over compact solids is their very low specific weight, in some cases as low as 10% that of water [1]. But cellular structures also possess unique properties that cannot be achieved at all with most conventional materials. These include very large surface to volume ratios, an ability to deform to very large strains at almost constant stresses, high porosities, and very well connected pore spaces. These properties make cellular structures attractive for a wide range of applications, including lightweight panels in the construction industry [?], light high-performance materials for the sport and leisure industries [?], impact dampers in the automotive, chemical and aviation industries [?], heat exchangers and filters, and special functionality materials for bio-artificial applications [?].

To take advantage of the unique properties of OCCS it is essential to have a fundamental understanding of their mechanical behavior. This proved difficult, in particular for disordered (sometimes termed stochastic) cellular microstructures. Macroscopic cellular materials can be regarded at a basic level as frameworks of irregularly connected struts, each of which possessing a unique topology. The topology profoundly affects the bulk properties, but it has been unclear how to take this type of effects systematically into consideration in modelling of the macroscopic mechanical behaviour.

Two types of models are commonly in use (not without overlap): models based on regular lattice-like structures, for which the computation of stresses is relatively straightforward, and models based on the replacement of the cellular structure by an equivalent continuous elastic medium. The first approach and its offshoots (e.g. perturbation of structural characteristics about the lattice structure) suffer from the shortcoming that they are valid only for ordered structures and their validity quickly deteriorates with the introduction of disorder. This limits the applicability of this approach because ordered structures are expensive to manufacture and therefore comprise a small fraction of existing and potential OCCS applications. The second type of models requires knowledge of effective elastic, or elasto-plastic, constitutive properties which need to be obtained independently, using theoretical, empirical, or experimental means. The conventional methods of coarse-graining and homogenization of such constitutive properties [2] [3] [4] [5] [6] are seriously limited in disordered cellular structures and a first-principles theoretical approach for their systematic derivation is yet to be developed. Consequently, this approach requires fresh measurements and parameter fitting for every material and microstructural characteristics, an inefficient and expensive process.

A landmark in the field is the work of Gibson and Ashby [7], who used scaling arguments to deduce power-law relations between the density and bulk elastic behaviour. The significance of that work is in that it relates macro-properties to microscopic structural information,

albeit very particular, and therefore that it bridges across the scales. It is, however, limited by its assumptions and simplifications. First, the material density is not sufficient as a parameter to characterise the microstructure; different topologies with the same density can have drastically different mechanical behaviour. Second, the derivation assumes periodic order on the cellular level and non-perturbative effects of disorder are not taken into consideration. Third, the macroscopic properties are presumed to hold on a sufficiently large lengthscale where the coarse-grained material is effectively homogeneous. It has become increasingly evident that the lack of a first-principles theory for relating between microstructures of generally disordered cellular solids and the stress field is a major barrier for progress on both the theoretical and the technological fronts.

It is important to note that underlying all the work in this field is the conventional wisdom that to model stresses in such materials all that is required is an adequate description of global, or bulk, elastic properties. This paper challenges this idea. The contention here is that this is a misguided premise whose addressing holds the key to resolving many of the difficulties in the field. One aim here is to demonstrate that the conventional approach misses an essential property of OCCS that can form the basis of a completely new approach. The key is a recent observation that the skeletons of most natural and man-made OCCS are statically determinate (SD) [8]. As will be shown, this observation has fundamental ramifications which could challenge the very paradigm underlying the current modelling of these materials. The discussion in this paper will focus, for clarity, on skeletal OCCS (SOCCS), namely, where the struts are infinitely thin compared to their length. However, it is possible to extend the theory developed here to cellular and porous solids comprising of struts of finite thickness and this will be done in a forthcoming report.

A stable structure is SD when all the forces that its components exert on one another can be determined from statics alone, namely, from the conditions of force and torque balance. Such structures have been studied more than a century ago by luminaries, such as Maxwell [9] and Levy [10]. Many engineering departments offer courses on SD systems and so it may appear that we know all there is to know about them. Curiously, this is not the case. The literature on SD systems commonly addresses structures with taxonomic numbers components, mainly because this makes feasible the computation of the forces between components. But how are we to model SD structures with large numbers of components so that the determination the individual forces is not practical even computationally? The best description is of course in terms of continuous functions. Furthermore, a continuous description is the only practical approach if the components of a SD structure are much smaller than the macroscopic scale of interest. Such a continuous description does not exist and its formulation is a fundamental issue in general.

This problem is not only academic. As will be shown below, OCCS are SD, and such structures consist of such large numbers of struts that numerical computations of strut forces are impractical. Since, for most purposes we are interested in the behaviour of these materials on lengthscales that are much larger than the typical size of a cell then it is imperative to have a continuous stress field description.

At this stage the reader may be wondering whether the author has missed a century and a half of development of elasticity theory (ET), which indeed deals with continuous stresses. The author has not. The problem with the current paradigm, of which ET is the flagship, is that it leads to a conundrum. While stresses in SD systems can be determined regardless of the force-deformation relations, ET requires independent knowledge of the equivalent stress-strain relations. We shall review the argument that leads to this conclusion in more detail below. This means that elasticity theory is in fact inadequate for these materials.

Thus, the central problem in the field of cellular solids is more fundamental than finding a consistent method of homogenisation of the elastic properties. It is the identification of the very equations that govern the continuous stress field in the first place.

This problem has been addressed recently for two-dimensional skeletal OCCS (SOCCS) [8], following an earlier development in granular packings [11] [12]. The primary aim of this paper is the solution of the new stress equations for planar SOCCS and an analysis of the solutions for general boundary conditions.

The paper is constructed as follows. Section II reviews the argument that disordered SOCCS are SD in two and in three space dimensions. Section III summarises the derivation of the constitutive relation that closes the field equations in planar cellular solids. In section IV the coupled field equations are manipulated to give decoupled equations for the components of the stress tensor. In section V these equations are solved to lowest order in the gradients of a fluctuating geometric field and the Green function for an infinite medium is derived. The solution is analysed in section VI and it is demonstrated that these correspond to nonuniform stresses propagating along characteristic lines, giving rise to preferential arches that support local loading. Section VII concludes with a discussion of the results, their implications, extension to higher order solutions, and a proposal for a formalism that extends the applicability of the theory to non-skeletal OCCS.

II. Static determinacy

Loading an OCCS by forces on its boundaries gives rise to internal forces between neighbouring vertices of the cellular structure. As discussed in the introduction, the first aim is to describe these forces in terms of a continuous stress field and then to find the continuous equations

that govern the field. Whatever the exact definition of the stress, in mechanical equilibrium it must satisfy force and torque balance conditions:

$$\partial_i \sigma_{ij} = g_j \quad \text{(Force balance)} \tag{1}$$

and

$$\sigma_{ij} = \sigma_{ji} \quad \text{(Torque balance)}\ , \tag{2}$$

where ∂_i is a derivative with respect to a spatial coordinate, $i = x, y, z$, $\vec{g}$ is an external force field (we ignore body forces here without loss of generality) and we adopt the convention of summation over repeated indices. In d dimensions ($d = 2, 3$) eq. (1) is a vector equation representing d components whilst eq. (2) is a pseudo-vector representing $d(d-1)/2$ components, corresponding to the number of axes of rotation. Thus, eqs. (1) and (2) give in total $d(d+1)/2$ equations. This number is $d(d-1)/2$ short of the $d \times d$ conditions needed to determine the d^2 components of the stress tensor. It follows that this many equations are required to close the set of field equations. These 'missing equations' must involve constitutive information. To place the issue in context, let us recall the paradigmatic approach. It follows Saint Venant's [13] postulation, where one imposes compatibility conditions on the medium, namely, that the material remain contiuous under small deformations. These conditions relate gradients of the deformation to one another. Another set of relations is then invoked between the deformation and the stress fields; the well known stress-strain relations. Thus, underlying the conventional approach is a fundamental conceptual premise that to determine stresses one must resort to information about the way that the material deforms under these very stresses. It is this basic premise of the necessity of coupling between the stress and the strain fields that needs to be questioned for cellular materials in particular and for macroscopic SD structures in general.

There are many situations where we take for granted that forces can be determined without reference to deformation-related data. For example, pressing a stick against a wall we expect that in mechanical equilibrium the longitudonal force is the same all along the stick and it is equal to the force that presses on it. We expect this force to be transferred to the wall *regardless* of the stick's deformation or its elastic properties. A less trivial, but widely familiar, example is that of a ladder resting on a rough floor and leaning against a smooth wall. In this textbook exercise it is possible to calculate the forces that the ladder applies on the wall and on the floor in terms of its weight, again regardless of the ladder's elastic properties; this structure is SD. In general, the key to being in a SD state is that the number of independent force components requiring determination is equal to the number of independent equations that can be constructed for them. In the ladder example the problem is effectively two-dimensional; there are three balance equations, two of force and one of torque moment, and they are sufficient to determine the two force components against the floor and the one force component against the wall. If we let the wall be rough as well then the problem becomes statically indeterminate because this adds an unknown tangetial force component along the wall while the number of available equations does not change.

In principle, there is no limit to the number of components of a SD structure. In particular, the number of intercellular forces in macroscopic cellular materials can range from 10^4 to 10^{10}. The following briefly reviews the arguments that lead to the conclusion that open-cell cellular structures are by and large SD both in two and in three-dimensions. Although the discussion is presented for infinite systems, the extension to finite systems is not difficult and will be presented elsewhere [14].

Disordered structure in two dimensions

A planar disordered trivalent SOCCS consist of an arbitrary assembly of struts in the plane connecting rigidly at N vertices ($N \gg 1$), with three struts meeting at every vertex. The struts are not limited to being straight. The structure is presumed to be in mechanical equilibrium under an arbitrary set of external forces acting on its boundary vertices. The loading generates forces along the struts which we term strut tensions although they may be either tensile or compressive. Due to the rigidity of the connections at the vertices the loading also generates torque moments around the vertices. These moments can be represented by an effective force acting at the mid point of every strut perpendicularly to the strut tension. We regard these two forces as the components of a strut force, e.g. $\vec{f}_{vv'}$ in figure 1b.

To find out whether the structure is statically determinate we need to compare the number of unknown force components to the number of equations avaliable to determine them. Let us enclose every vertex by an imaginary triangle, whose corners lie at the midpoints of the strut that meet at the vertex (see figure 1a). Two neighbouring triangles, say around vertices v and v', are presumed to connect rigidly at their contact points and exert on one another a vector force $\vec{f}'_{vv'}$. The triangles form a rigid framework in the plane, enclosing polygons that are contained within the cells of the original structure. Postulating that the network of triangles is in mechanical equilibrium under the same set of forces as the original structure, $\vec{f}_{vv'} = \vec{f}'_{vv'}$, all we need to determine are the forces that the triangles exert on one another through their contacts. Since each contact lies at the mid-point of a given strut then the contact force can be identified with a specific strut and we term it a strut force. The strut force can be decomposed into two components: the strut tension and a force perpendicular to it. The latter puts a torque moment on the vertices at the two ends of the strut (see figure 1b). Thus constructed, the framework of triangles carries the same field of discrete forces as the original structure and would therefore support an

identical stress field.

How many forces are there to determine? Since every strut ends at two vertices and every vertex connects three struts then, up to boundary corrections of order $\sqrt{N}$ (which can be included explicitly in the argument and do not affect the generality of this argument [14]), there are $3N/2$ struts. With two components per strut force, this gives altogether $3N$ unknowns.

How many equations can we construct to determine these forces? Each of the N triangles is in mechanical equilibrium and therefore it satisfies three balance equations; two of force and one of torque moment. This gives in total $3N$ equations; exactly enough to determine all the unknowns. It follows that planar trivalent SOCCS are SD.

In structures where struts meet at angles of exactly $2\pi/3$ the above argument does not apply directly, but a slightly modified version of it can be constructed, leading to the conclusion that structures of this type are also SD [8]. Note that the symmetry around the vertices does not give rise to an ordered structure. The disorder rises from both a statistical distribution of strut curvatures and a distribution of strut lengths.

Disordered structures in three dimensions

Before extending the above discussion to three dimensions, let us make a few general observations. A disordered three-dimensional SOCCS consists of an arbitrary assembly of struts in space, connecting rigidly at N vertices. The following discussion presumes large systems ($N \gg 1$), in which case boundary corrections can be ignored. Nevertheless, here too the arguments apply equally well to finite systems [14]. In quadrivalent structures four struts, which need not be straight, meet at every vertex and each strut is shared by two vertices. This means that in a structure of N vertices there are $2N$ struts in total; two per vertex.

Let us now imagine that we can view the interior of a particular cell c from a point within it. The 'surface' of the cell, which from this vantage point is visible as a planetarium, comprises of n_v^c vertices connected by n_t^c struts, all of which making n_f^c facets. Including cell c, the vertices are common to four cells, the struts are common to three cells, and each facet is shared between two cells. Thus, if we sum over all cells we have

$$\begin{aligned} \langle n_v \rangle &\equiv \frac{1}{N_c}\sum_c n_v^c = 4\frac{N}{N_c} \\ \langle n_t \rangle &\equiv \frac{1}{N_c}\sum_c n_t^c = 3\times\frac{2N}{N_c} = 6\frac{N}{N_c} \\ \langle n_f \rangle &\equiv \frac{1}{N_c}\sum_c n_f^c \ , \end{aligned} \tag{3}$$

where angular brackets denote an average per cell. The cell surface has the topology of a sphere and therefore the vertices, struts and facets satisfy Euler's relaion [28]

$$n_v^c - n_t^c + n_f^c = 2 \ . \tag{4}$$

Summing Euler's relation over all cells and using (3) gives

$$\sum_c \left(n_v^c - n_t^c + n_f^c\right) = 4N - 6N + \langle n_f\rangle N_c = 2N_c \ . \tag{5}$$

From this we get that

$$\langle n_f \rangle = 2\left(1 + \frac{N}{N_c}\right) \ . \tag{6}$$

and, using relations (3) and (4) in (6), further gives that

$$\begin{aligned} \langle n_f \rangle &= 2 + \frac{1}{2}\langle n_v \rangle \\ \langle n_f \rangle &= 2 + \frac{1}{3}\langle n_t \rangle \ . \end{aligned} \tag{7}$$

We can also obtain relations for individual cells rather than for averages over all cells. The n_v^c vertices on the surface of cell c are each connected to three surface struts and each surface strut is shared by exactly two of these vertices, which gives that

$$n_t^c = \frac{3}{2}n_v^c \ . \tag{8}$$

Substituting this into Euler's law (4) we see that every cell must satisfy the following relations

$$\begin{aligned} n_f^c &= 2 + \frac{1}{2}n_v^c \\ n_f^c &= 2 + \frac{1}{3}n_t^c \ . \end{aligned} \tag{9}$$

Upon averaging over cells these relations indeed recover relations (7). Another set of relations that can be obtained is the following. Indexing the facets of cell c by f and denoting the number of sides of this facet by m_f^c, we can sum over all the facets on the surface of a particular cell to find

$$\sum_f m_f^c = 2n_t^c = 3n_v^c \ . \tag{10}$$

This is because by counting over the facets each strut is counted twice and each vertex is counted three times. If we now further sum over cells we get that per cell

$$\frac{1}{N_c}\sum_c\sum_f m_f^c = 12\frac{N}{N_c} \ . \tag{11}$$

We can also sum directly over the cell facets to obtain

$$\frac{1}{N_c}\sum_c n_f^c = 2\langle n_f\rangle \ . \tag{12}$$

Not all the relations above are independent, of course. It should be emphasized that, unlike in two dimensions, this analysis cannot give the value of the ratio N_c/N.

This poses no problem in the demonstration of static determinacy of arbitrary disordered structures, but it does prevent such a demonstration for symmetric structures, in analogy to the symmetric structures discussed in two dimensions. Consequently, we shall not discuss here any further symmetric structures in three dimensions.

However, it should be commented that the mean number of facets around a cell, $\langle n_f \rangle$, can be argued to be the same as the mean coordination number of packings of *smooth* particles. It has been conjectured already by Newton that this quantity is twelve for spheres. This is the minimum number for mechanical equilibrium in such systems [**?**] and experimental and numerical measurements in polydisperse packings suggest that this value is more likely to be around fourteen. Substituting these values into relation (6) we get an estimate of N_c in terms of N

$$\frac{N}{6} < N_c < \frac{N}{5} \ . \tag{13}$$

Back to the issue of static determinacy. Consider a SOCCS in mechanical equilibrium under an arbitrary set of external forces acting on its boundary vertices. Each bulk vertex is connected by struts to four nearest-neighbour vertices and the angles that the struts make at the vertex are presumed arbitrary. The loading generates forces along the struts (positive or negative strut tensions) and torques around the vertices.

To examine the static determinacy of the structure we resort again to an auxiliary construction. Connect the midpoints of the struts around every vertex by imaginary rigid struts, forming a tetrahedron around the vertex. Neighbouring tetrahedra, such as the two around vertices v and v', make contact at the midpoint of the strut between the vertices and the contact is presumed to be rigid. The framework of tetrahedra is rigid and we postulate that it is in mechanical equilibrium under the same set of forces as the original structure, We need then to determine is in mechanical equilibrium under the same set of forces as the original structure,the forces that the tetrahedra exert on one another through their contacts. As in two dimensions, every contact is associated with a strut, which gives that there is one force vector to determine per strut. The components of such a force are defined to be $\vec{T}_{vv'}$ along the strut and $\vec{F}_{vv'}$ in the plane perpendicular to $\vec{T}_{vv'}$. Thus defined, the framework of tetrahedra transmits the same forces as the original SOCCS. Since there are $2N$ struts, each carrying a three-dimensional force vector, then there are $6N$ force components to determine.

How many equations can we construct to determine them? Each of the N tetrahedra is in mechanical equilibrium and it therefore satisfies three equations of force balance and three equations of torque balance, one around each axis of rotation. This gives $6N$ equations in total, exactly the right number to determine the $6N$ unknowns. Conclusion: three-dimensional quadrivalent SOCCS are SD.

III. The constitutive equation in two dimensions

The conclusion that disordered SOCCS are SD both in two and in three dimensions has one very significant ramification - conventional elasticity theory cannot be used to determine the stress field. The reason is that the continuous field equations must arise from balance conditions alone and therefore, as mentioned in the introduction, stress-strain relations are redundant. This problem occurs also in the field of granular systems [15], where the search for replacement constitutive relations, which can close the field equations, has attracted much attention in recent years [15]. Empirical stress-structure relations, proposed by the Edinburgh group [16], were followed by a mean-field statistical approach [17], but very recently a first-principles theory was developed for two-dimensional systems [11]. In two dimensions there is only one constitutive equation, which the theory derived on the scale of a few grains. This equation was then coarse-grained to the macroscale in [12]. The following is a brief summary of the theory.

The derivation of the constitutive equation goes in several steps: First, the force vectors at the strut midpoints are described in terms of cell forces that satisfy the force balance conditions. Second, the cell forces are coarse-grained into a continuous force field. Third, the stress is described as a function of the continuous field. Fourth, the torque balance condition is applied on the stress, which yields the continuous constitutive equation. The fifth and last step consists of coarse-graining of the equation to macroscopic lengthscales.

Step 1:

Starting from the construction shown in figure 1, let us introduce several definitions. First, we assign directions to the edges of the triangles around the vertices so that every triangle circulates its vertex in the clockwise direction. The centroid of triangle v is defined as the mean position of its three corners. Each edge vector $\vec{r}$ lies within a particular cell c and can be uniquely indexed $\vec{r}_{cv}$ (see figure 2). These vectors form a network; the contact network. Let us further define the centroids of the cells as the mean positions of the midpoints of the strutes around them. Finally, we define vectors $\vec{R}_{cv}$ that extend from the centroid of triangle v to the centroid of a neighbor cell c. These vectors form another network; the centroid network. The centroid and the contact networks are self-dual; an edge of one intersects an edge of the other.

Let us parameterize the strut forces in terms of cell forces. Each cell is assigned a two-dimensional force vector $\vec{f}_c$ located at the centroid of the cell. The parameterization takes advantage of the fact that the contact point between v and v' also lies at the border between two adjacent cells l and m (see figure 2). The force that triangle v exerts on triangle v' in terms of the cell forces is

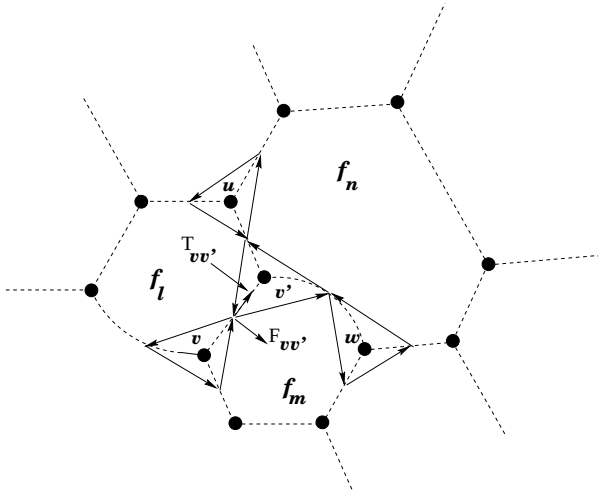

FIG. 1. [*Figure 3*] The parameterization of the strut forces around vertex v' in terms of its neighboring cell forces. $\vec{f}_{vv'} = \vec{f}_l - \vec{f}_m$ is the force that triangle v exerts on triangle v'. The sign convention is that $\vec{f}_l$ is positive because the directed loop formed by the vectors $\vec{r}$ around cell l is *toward* triangle v' and vice versa for cell m.

$$\vec{f}_{vv'} = \vec{f}_l - \vec{f}_m \tag{14}$$

The sign convention in eq. (14) is such that $\vec{f}_c$ is positive when the directed loop formed by the vectors $\vec{r}_{cv}$ around cell c is from v toward v' at the vv'-contact and vice versa. This parameterization shifts the problem from the determination of the strut forces to the determination of the fictitious cell forces and this has several advantages: (i) By construction, $\vec{f}_{v'v} = \vec{f}_m - \vec{f}_l = -\vec{f}_{vv'}$ and therefore the cell forces automatically satisfy Newton's third law everywhere. (ii) A summation over the forces that act on vertex v (see figure 3) gives

$$\sum_{v'} \vec{f}_{vv'} = (\vec{f}_l - \vec{f}_m) + (\vec{f}_m - \vec{f}_n) + (\vec{f}_n - \vec{f}_l) \equiv 0 \ , \tag{15}$$

which means that the cell forces balance automatically on every triangle. (iii) There are $N/2$ cells and one force per cell. Therefore the number of unknowns has been reduced from $3N$ to N. This amounts to a coarse-graining of the force field by a factor of three, which, since the structure is SD, must be matched by a reduction in the number of equations. This reduction is because the loop forces already satisfy the force balance equations and only one equation of torque balance per triangle remains, leaving in total N equations with N unknowns.

Step 2:

Next, the discrete field of cell forces is continued into a continuous field, $\vec{\mathcal{F}}$. This was done in reference [11] using the following interpolation scheme. The three cell forces around every vertex are interpolated piecewise linearly, as shown in figure 4, giving a continuous vector function. Each of the two components of this function, $\mathcal{F}^\alpha$ defines a surface comprising of adjacent planar triangles.

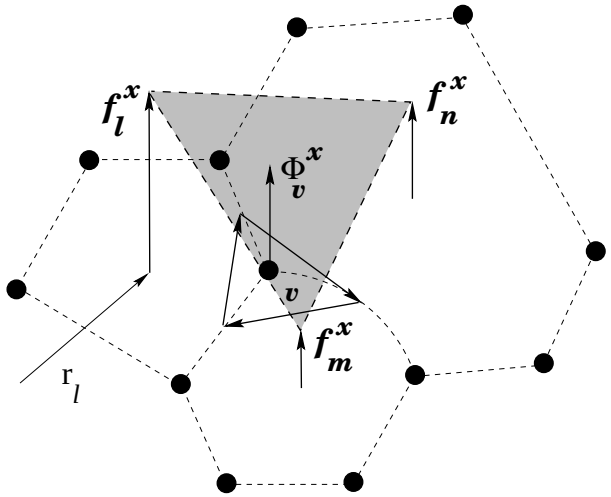

FIG. 2. [*Figure 4*] The discrete field of cell forces is made into a continuous force field $\vec{\mathcal{F}}$ by constructing a piecewise linear triangular surface around each vertex. An example of such a surface for the x-component of the cell forces around vertex v is shown. The corners of the triangular surface are above the centroids of the cells l, m and n that are located at $\vec{r}_l$, $\vec{r}_m$, and $\vec{r}_n$. The heights of the planar surface at these points are, respectively, $\vec{f}_l^x$, $\vec{f}_m^x$, and $\vec{f}_n^x$. The value of the surface above the vertex gives $\vec{\mathcal{F}}_v$. In terms of the continuous field, the value of any cell force $\vec{f}_c$, immediately neighboring vertex v, is exactly $\vec{f}_c = \vec{\mathcal{F}} + \vec{R}_{cv} \cdot \vec{\nabla}\vec{\mathcal{F}}$.

The corners of each such a triangle are located above the centroids of cells l, m, and n, and the heights of these corners are, respectively, f_l^α, f_m^α, and f_n^α (see illustration in figure 4 for $\alpha = x$). The value of the continuous force field at the vertex v is $\vec{\mathcal{F}}_v$, the weighted mean of the cell forces around it. In terms of the continuous field $\vec{\mathcal{F}}$, the cell forces around v are

$$\vec{f}_c = \vec{\mathcal{F}}_v + \vec{R}_{cv} \cdot \vec{\nabla}\vec{\mathcal{F}} \ . \tag{16}$$

This relation is exact in the sense that no higher order gradients are involved. This is a clear advantage of the linear continuation scheme; in the context of other schemes eq. (16) can only be regraded as a first-order approximation to a series expansion.

Step 3:

The force moment on triangle v can now be expressed in terms of the vectors $\vec{r}_{cv}$ and forces $\vec{f}_c$

$$S_v^{ij} = \sum_c r_{cv}^{\ i} f_c^{\ j} \ , \tag{17}$$

where the sum runs over all the cells surrounding vertex v. Substituting (16) into (17) and noting that around a vertex $\sum_c \vec{r}_{cv} = 0$ gives

$$S_v^{ij} = \sum_c r_{cv}^{\ i} R_{cv}^k \partial_k \mathcal{F}^j = \left(\hat{C}_v \cdot \vec{\nabla}\vec{\mathcal{F}}\right)^{ij} \ , \tag{18}$$

where a new geometric tensor has been defined

$$\hat{C}_v^{ij} = \sum_c r_{cv}^i R_{cv}^j \ . \tag{19}$$

This tensor characterizes the local geometry around vertex v and plays an central role in the analysis. It is convenient to separate this tensor into its antisymmetric, $A_v\hat{\epsilon} = (\hat{C}_v - \hat{C}_v^T)/2$, and symmetric, $\hat{P}_v = (\hat{C}_v + \hat{C}_v^T)/2$, parts. Here A_v is a prefactor and $\hat{\epsilon} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the $\pi/2$ rotation tensor. Both these parts have direct geometrical interpretations: $A_v = |\vec{r}_{cv} \times \vec{R}_{cv}|/2$ is the area associated with vertex v that is shown shaded in figure 5. The entire system can be conveniently tesselated and tiled by these irregular non-convex hexagons, namely, $A_{sys} = \sum_v A_v$. This feature is useful for calculations of effective volumes and configurational entropy [19]. For the geometrical interpretation of the symmetric part note that if each of the three pairs of vectors appearing in $\vec{r}_{cv}$ and $\vec{R}_{cv}$ in expression (19) were perpendicular around the triangle then $\hat{P}_v$ vanishes. A net finite value means a tilt in either the clockwise or anticlockwise directions, namely, a net rotation relative to the immediate cells. This net rotation clearly must vanish when averaged over the entire structure, which can be shown by representing the vectors $\vec{r}_{cv}$ and $\vec{R}_{cv}$ in expression (19) in terms of the vectors $\vec{s}_{cv}$ and $\vec{t}_{cv}$ shown in figure 6. Through an algebraic manipulation of the vectors surrounding the quadrilateral whose diagonals are $\vec{r}_{cv}$ and $\vec{R}_{cv}$, it can be shown that

$$\hat{P}_v = \frac{1}{2}\sum_c \left(\vec{s}_{cv}\vec{s}_{cv} - \vec{t}_{cv}\vec{t}_{cv}\right) \ . \tag{20}$$

This representation makes it possible to see that when $\hat{P}_v$ is summed over any region within the system, the contribution from cells fully enclosed inside the region cancel out and the only non-vanishing contribution comes from boundary cells. Thus, the average of the tensor $\hat{P}$ over the entire system vanishes at least as fast as $1/\sqrt{N}$. It follows that, this tensor, which is a measure of the total *rotational* disorder around vertex v, fluctuates as a function of position in the plane around a zero mean. This is an important observation to which we shall return later.

Step 4:

Let us now impose the torque balance condition on the continuous stress. This requires a preparatory calculation. Consider a small region Γ consisting of several cells, whose boundary is $\partial\Gamma$. The mean stress inside Γ is

$$\hat{\sigma}_\Gamma = \frac{\langle \hat{S}_v \rangle_\Gamma}{A_\Gamma} = \frac{\langle \hat{C}_v \cdot \vec{\nabla}\vec{\mathcal{F}} \rangle_\Gamma}{A_\Gamma} \tag{21}$$

where, for clarity, $\langle \ldots \rangle_\Gamma$ stands for a sum over the region Γ and A_Γ is its area. It has been shown in [11] that over such a region

$$\langle \hat{C}_v \cdot \vec{\nabla}\vec{\mathcal{F}} \rangle_\Gamma = \langle \hat{C}_v \rangle_\Gamma \cdot \langle \vec{\nabla}\vec{\mathcal{F}} \rangle_{\partial\Gamma} \tag{22}$$

where $\langle \ldots \rangle_{\partial\Gamma}$ stands for an average over the boundary of Γ. Combining (21) and (22) we have

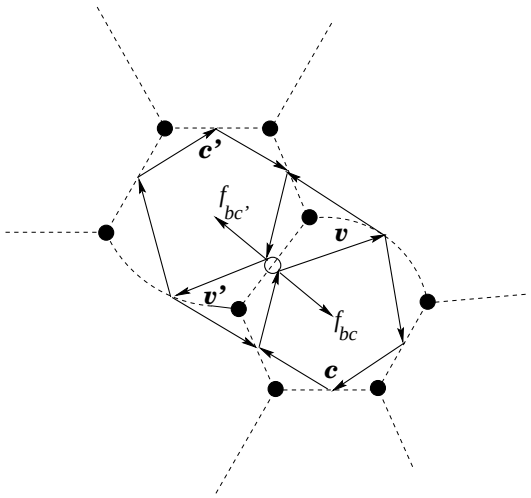

FIG. 3. Making all the grains rough and inserting a vanishingly small ball bearing (white circle) between touching grains at the contact maintains the original intergranular forces.

$$\langle \vec{\nabla}\vec{\mathcal{F}} \rangle_{\partial\Gamma} = \langle \hat{C}_v \rangle_\Gamma^{-1} A_\Gamma \hat{\sigma}_\Gamma = \hat{\epsilon}^{-1}\hat{\sigma}_\Gamma \tag{23}$$

where the last equality uses the fact that the average of $\langle \hat{C}_v \rangle_\Gamma = A_\Gamma[\hat{\epsilon} + \langle P \rangle_\Gamma]$ and that the average $\langle P \rangle$ vanishes faster than $1/\sqrt{N}$.

We can now apply the condition of torque balance to a small region in the system

$$\mathcal{A}\left\{\hat{S}\right\} = \mathcal{A}\left\{\hat{C} \cdot \vec{\nabla}\vec{\mathcal{F}}\right\} = \mathcal{A}\left\{\hat{C}\hat{\epsilon}^{-1}\langle\hat{\sigma}\rangle\right\} = 0 \tag{24}$$

where $\mathcal{A}\{\ldots\}$ stands for the antisymmetric part of $\{\ldots\}$. The angular brackets and the subscript v have been dropped for brevity, under the understanding that $\hat{S}$ and $\hat{\sigma}$ now represent the coarse-grained stress field on the scale of basic regions Γ of a small size. Writing $\hat{C}$ explicitly we have

$$\mathcal{A}\left\{\hat{P}\hat{\epsilon}^{-1}\hat{\sigma} + A_\Gamma\hat{\sigma}\right\} = 0 \ . \tag{25}$$

The second term on the left, which comes from the antisymmetric part of $\hat{C}$, is exactly the global torque balance condition, eq. (2). The first term is a *local* torque balance condition that reads explicitly

$$\mathcal{A}\left\{\hat{P}\hat{\epsilon}^{-1}\hat{\sigma}\right\} = p_{xx}\sigma_{yy} + p_{yy}\sigma_{xx} - 2p_{xy}\sigma_{xy} = 0 \tag{26}$$

This is the missing constitutive equation of reference [11]. Defining $\hat{Q} = \hat{\epsilon}\hat{P}\hat{\epsilon}^{-1}$, this equation can be rewritten more compactly as [20]

$$q_{ij}\sigma_{ij} = 0 \quad \text{or} \quad \hat{Q} : \hat{\sigma} = 0 \tag{27}$$

Like the tensor $\hat{P}$, $\hat{Q}$ is symmetric and characterizes local rotational disorder. The constitutive equation (26) couples the stress to local fluctuations in the geometry of the cellular structure and completes the set of equations for the stress field in planar systems.

Step 5:

In eq. (26) the parameters p_{ij} represent an average over a few cells within a small region Γ and coarse-graining is required to get to macroscopically relevant lengthscales. As observed in ref [11], but this is hindered by the vanishing of the area average of the p_{ij}. This difficulty has been resolved recently [12]. The details of the coarse-graining procedure are not of major consequence to the thrust of this paper. However, two of its consequences are worthwhile to recall. One is that the macroscopic equation has the same form as (30), with the stress field having the same interpretation but with the constitutive parameters p_{ij} averaged roughly over every *other vertex*. Another is that the volume average of the p_{ij}s is finite because most values have the same sign, making it possible to use *conventional* method to continue the coarse-graining to ever larger scales.

IV. Decoupling the field equations

The two-dimensional field equations can now be written explicitly

$$\partial_x \sigma_{xx} + \partial_y \sigma_{xy} = g_1 \tag{28}$$
$$\partial_x \sigma_{xy} + \partial_y \sigma_{yy} = g_2 \tag{29}$$
$$p_{yy}\sigma_{xx} + p_{xx}\sigma_{yy} = 2p_{xy}\sigma_{xy} \ , \tag{30}$$

where the stress tensor has already been taken to be symmetric, implementing the condition of global torque balance. The continuous equations are valid on a scale above the cellular level but much smaller than the size of the system.

Eqs. (28)-(30) are coupled and to solve for the stress field we first need to decouple them. Let us assume, for clarity, that $p_{ij} \neq 0$ everywhere. The special cases when one of these parameters vanishes are simpler to analyse and can be solved exlicitly. For example, the equation for σ_{xy} is obtained as follows. Substite σ_{yy} from eq. (30) into eq. (28), which gives

$$\partial_x \sigma_{xy} + \partial_y \left(2r_{xy}\sigma_{xy} - r_{yy}\sigma_{xx}\right) = g_1 \ , \tag{31}$$

where $r_{ij} = p_{ij}/p_{xx}$. From eq. (28) we have that

$$\sigma_{xx} = \int \left(g_1 - \partial_y \sigma_{xy}\right) dx + \Phi(y) \ , \tag{32}$$

where $\Phi(y)$ is an arbitrary function of y that arises as a constant of integration. Substitution of expression (29) into eq. (31) now yields an integro-differential equation for σ_{xy}

$$\begin{aligned}
&\left(p_{xx}\partial_{xx} + 2p_{xy}\partial_{xy} + p_{yy}\partial_{yy}\right)\sigma_{xy} + \\
&p_{xx}\left(\partial_{xy}r_{yy} + \partial_y r_{yy}\partial_x + \partial_x r_{yy}\partial_y\right)\int \partial_y \sigma_{xy} dx = \\
&\left(p_{xx}\partial_x g_2 + p_{yy}\partial_y g_1\right) + \\
&p_{xx}\left(\partial_{xy}r_{yy} + \partial_y r_{yy}\partial_x + \partial_x r_{yy}\partial_y\right)\left(\phi(y) + \int g_1 dx\right) \ .
\end{aligned} \tag{33}$$

In spite of its formidable appearance, this equation can be solved by an expansion method [14]. Since the consitutive parameters p_{ij} fluctuate in the plane then the coarse-graining smooths out the fluctuations whose gradients, $\lambda_{ijk} \equiv \partial_i p_{jk}$, should become smaller with increasing of the lengthscale. Thus, in macroscopic structures these gradients are expected be smaller than the gradients of the stress field. This makes it possible to expand in these quantities and obtain the solution of eq. (33) order by order [14]. Here only the lowest order solution of this expansion is discussed where the λ_{ijk}s are neglected altogether. To this order eq. (33) reduces to

$$\left(p_{xx}\partial_{xx} + 2p_{xy}\partial_{xy} + p_{yy}\partial_{yy}\right)\sigma_{xy} = p_{xx}\partial_x g_2 + p_{yy}\partial_y g_1 \equiv f_{xy}$$

or

$$\left(p_{nm}\partial_{nm}\right)\sigma_{xy} = f_{xy} \ . \tag{34}$$

Corresponding derivations for the components σ_{xx} and σ_{yy} to this order yield similar equations;

$$\left(p_{nm}\partial_{nm}\right)\sigma_{ij} = f_{ij} \quad \text{or} \quad \hat{P} : (\nabla \otimes \nabla)\,\hat{\sigma} = \hat{f} \ . \tag{35}$$

The elements of the tensor $\hat{f}$, the source terms f_{ij}, are functions of the geometric parameters and of the gradients of the external field $\vec{g}$;

$$\begin{aligned}
f_{xy} &= p_{yy}\partial_y g_x + p_{xx}\partial_x g_y \\
f_{xx} &= \left(p_{xx}\partial_x + 2p_{xy}\partial_y\right) g_x - p_{xx}\partial_y g_y \\
f_{yy} &= \left(p_{yy}\partial_y + 2p_{xy}\partial_x\right) g_y - p_{yy}\partial_x g_x
\end{aligned}$$

Thus, the field equations have been decoupled into one equation for each stress component. The equations have the convenient property that they follow an identical differential form, $\hat{P} : (\nabla \otimes \nabla) = p_{mn}\partial_{mn}$ operating on the stress components, and they differ only in the source terms f_{ij}. This form allows explicit solutions.

V. Solution of the field equations

We first observe that if $\det\hat{P} > 0$ (< 0) then the eqs. are elliptic (hyperbolic). To determine which it is recall that on the vertex level

$$p_{ij}^v = \frac{1}{2}\sum_{c=1}^{3}\left(r_i^{cv}R_j^{cv} + r_j^{cv}R_i^{cv}\right) \equiv \sum_{c=1}^{3} p_{ij}^{cv} \tag{36}$$

and that $\det\hat{P}^{cv} = -(A^{cv})^2 < 0$, where A^{cv} is the area of the quadrilateral spanned by the diagonals $\vec{r}^{cv}$ and $\vec{R}^{cv}$, shown shaded in figure X. After some algebra we obtain that, in terms of the quadrilaterals around vertex v,

$$S \equiv \sqrt{-\det\hat{P}^v} = \sqrt{(A^v)^2 - \sum_{c,c'}(\vec{r}^{cv} \times \vec{r}^{c'v}) \cdot (\vec{R}^{cv} \times \vec{R}^{c'v})} \tag{37}$$

where $A^v = \sum_l A^{cv}$. A careful examination of this expression shows that on average over roughly every other

vertex, as done in the coarse-graining, $\det\hat{P} < 0$, leading to the conclusion that eqs. (35) are indeed hyperbolic on scales much larger than the cell size.

It is now convenient to transform the variables locally using

$$\binom{u}{v} = \frac{1}{S}\begin{pmatrix} S & 0 \\ -p_{xy} & p_{xx} \end{pmatrix} \cdot \binom{x}{y} \tag{38}$$

Expressing eqs. (35) in terms of the new variables transforms it into the convenient form

$$(\partial_{uu} - \partial_{vv})\,\sigma_{ij} = f_{ij} \ . \tag{39}$$

[ADD A FEW WORDS HERE]

VI. Analysis of the solution and force chains

The general solution of eq. (39) iin the variables u and v s a textbook exercise;

$$\sigma_{ij} = A^{+}_{ij}\phi(\eta) + A^{-}_{ij}\psi(\zeta) + B^{+}_{ij}\eta + B^{-}_{ij}\zeta + \tag{40}$$

$$\frac{1}{4}\int^{\eta}\int^{\zeta} f_{ij}(\eta',\zeta')d\eta' d\zeta' \ , \tag{41}$$

where $\eta = v - u$ and $\zeta = v + u$ are the characteristic curves of the solution,

$$\hat{A}^{\pm} = \begin{pmatrix} \frac{p_{xx}}{\alpha^{\pm}} & 1 \\ 1 & \frac{\alpha^{\pm}}{p_{xx}} \end{pmatrix} \quad \text{and} \quad \hat{B}^{\pm} = \begin{pmatrix} \frac{\alpha^{\mp}}{p_{yy}} & 1 \\ 1 & \frac{\alpha^{\pm}}{p_{xx}} \end{pmatrix} b^{\pm} \ ,$$

and $\alpha^{\pm} = p_{xy} \pm S$. The functions ϕ and ψ and the coefficients $b^{\pm}$ are arbitrary and need to be determined by the boundary data.

It is instructive to write the solution for the stress in a particular geometry. Consider an open-cell solid foam of arbitrary internal structure, occupying the half plane $x \geq 0$. The local microstructure is presumed known everywhere in the system in the sense that we know $p_{ij}(x,y)$. On the boundary $x = 0$ we impose the following boundary data: $\sigma_{ij}(x = 0, y) = U_{ij}(y)$ and $\partial_x(x = 0, y) = V_{ij}(y)$. The constitutive tensor $\hat{p}$ is presumed to be an arbitrary function of position, subject to the condition that $\det\hat{p} < 0$. For simplicity, let us assume that the value of the constitutive tensor is constant on the boundary independent of y, namely, $p_{ij}(x = 0, y) \equiv \pi_{ij}$ =constant. Most convenient is to carry out the analysis in the $v - u$ plane, for which the boundary data read

$$\sigma_{ij}\left(u = o, v = \frac{\pi_{xx}}{\Sigma}y\right) = U_{ij}\left(\frac{\Sigma}{\pi_{xx}}v\right) \equiv \Phi_{ij}(v)$$

$$\partial_u\sigma_{ij}\left(u = o, v = \frac{\pi_{xx}}{\Sigma}y\right) = V_{ij}\left(\frac{\Sigma}{\pi_{xx}}v\right) + \frac{\pi_{xy}}{\Sigma}U'_{ij}\left(u = o, v = \frac{\pi_{xx}}{\Sigma}y\right) \equiv \Psi_{ij}(v) \ , \tag{43}$$

where $\Sigma = -det\hat{\pi}$ and U'_{ij} is the derivative of U_{ij} with respect to its argument. In terms of these data the solution to eq. (39) in a constant external field $\vec{g}$ is

$$\sigma_{ij}\,(u,v) = \frac{1}{2}\left[\Phi_{ij}\,(v-u) + \Phi_{ij}\,(v+u)\right] + \frac{1}{2}\int_{\zeta}^{\eta}\Psi_{ij}(t)dt \ . \tag{44}$$

Transforming this solution back into the original $x - y$ coordinates gives

$$\sigma_{ij}\,(x,y) = \frac{1}{2}\left\{U_{ij}\left(\frac{\Sigma}{\pi_{xx}}\frac{p_{xy}y - \alpha^{+}x}{S}\right) + U_{ij}\left(\frac{\Sigma}{\pi_{xx}}\frac{p_{xy}y - \alpha^{-}x}{S} + \int_{\frac{p_{xy}y-\alpha^{+}x}{S}}^{\frac{p_{xy}y-\alpha^{-}x}{S}}\left[V_{ij}(t) + \frac{\pi_{xy}}{\Sigma}U'_{ij}(t)\right]dt \ . \right.\right. \tag{45}$$

Example: Consider a solid foam of arbitrary structure whose p_{ij} are known everywhere. The system is loaded on its boundary by a localised stress $\sigma_{ij} = \delta_{ix}\delta_{jx}T(0, 2d, y)$, where

$$T(c, 2d, y) \equiv \begin{cases} 1 & \text{c-d} \leq y \leq c + d; \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

is a hat-function of width d centered on c.

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VII. Conclusion, discussion and future directions

This paper addressed the problem of determining stresses in planar trivalent and three-dimensional quadrivalent SOCCS. The argument that these structures are statically determinate has been reviewed in detail. An important implication of static determinacy is that in mechanical equilibrium the stress field can be determined independently of the strain field at this state. This obviates elasticity theory for these materials due to the redundancy of stress-strain relations that it relies on. Instead, a new isostaticity theory is required, where the stress equations are closed by local constitutive stress-geometry relations. A first-principles such a theory has been described in two dimensions. The field equations were decoupled and solved to lowest order in the gradients of the constitutive parameters. The solutions propagate along characteristic lines that are determined by the local microstructure. These lines are proposed to correspond to force chains along which the forces are non-uniformly supported by the cellular structure.

Fundamental issues:

(i) Although the discussion here focused on comparison with elasticity, it should be remembered that *all* conventional theories of stress transmission are based on constitutive relations that link the stress to deformation. Elasticity employs strain while elastoplastic and viscoelastic theories are based on relations to strain rate. By

resorting to statics alone, the theory developed here is therefore outside the existing paradigm altogether. It has been shown by BB that the new formalism recovers the Airy stress function as the solution for the symmetric divergence-free stress field. The fact that this solution, which is usually obtained within conventional elasticity theory, can be derived without the use of any stress-strain relation clearly demonstrates that these are redundant for determining the stresses that develop in trivalent solid foams. This also means that there is a connection between the two approaches. To see why such a relation must exist suppose we apply an infinitesimally small strain to a solid foam, of the type discussed here, at the end of which the medium settles into the aforementioned isostatic state. Elasticity theory should be able to calculate the stress field just before this state is reached, while the new theory should be able to do so after all deformation has ceased. The stress field cannot change discontinuously at the end of the infinitesimal straining process and therefore both theories must yield the same answer in the limit when the strain goes to zero. This issue is currently under investigation by this author and preliminary results suggest that indeed there is a relation between the two theories.

(ii) As mentioned already, in the context of granular assemblies, when all the intergranular forces can be determined from statics alone the system is in a marginal rigidity state [21] [22], which is interpreted as a new state of solid matter. In granular assemblies, the basic condition for being in this state is that the average coordination number per grain is three for rough grains and six for smooth. The state of matter of any system can be determined by its response to external mechanical loading: Gases need to be held together by an external pressure, liquids flow under stress and solids first deform and then transmit the stress statically. The difference between the equations of stress transmission in marginally rigid assemblies and in conventional solids implies different responses to external loading and hence, by definition, that these are in different states of solid matter. In granular systems this new state lies between the fluid and the traditional solid states. Adding more contacts between grains necessitates additional constitutive information beyond the knowledge of the structure alone, while a reduction in the number of contacts takes the assembly outside mechanical equilibrium and renders it fluid. This feature of existence between two conventional states appears to have no immediate analog in solid foams. It is possible that such an analog can be found in the context of the rheology of foams close to the so-called jamming transition point. It is interesting to note that in granular systems there is a direct relation between the constitutive equation (27) and the dynamics of yield [23]. It would be interesting to try to extend this connection to the flow of foams.

(iii) The basic principle of force determination on the cellular level without resorting to the compliance of the cell walls does not appear to rely heavily on the two-dimensional nature of the system. In fact, it is possible to extend this argument to open-cell foams where each vertex is the meeting point of four edges and six walls, a rather ubiquitous class of structures in nature. This suggests that the stress field in such three-dimensional systems should also be independent of stress-strain relations and therefore that there is a larger class of solid foams that are at a state of marginal rigidity.

Practical issues:

(i) First and foremost, this theory will clearly make it possible to model disordered planar cellular materials to predict stress transmission, extending the current capabilities. In particular, the theory should make it straightforward to predict 'hot spots' of stress concentration in such systems and make useful connections between the occurrence of hot spots and the microstructure. This knowledge is expected to both improve prediction of failure in advanced materials and set new goals for the design of microstructural characteristics that prevent such vulnerability.

(ii) The formal mapping between granular assemblies and cellular systems can be used to transfer results between the two fields. Granular assemblies can only approach the marginal rigidity state under careful consolidation processes [21]. In contrast, it is quite common for the dynamics of formation of foams to give rise to triple junctions and so drive the structure naturally to this state. Morever, once a foam has solidified its structure is set, while granular assemblies are fragile by the very nature of this state. This difference means that most measurements in granular assemblies are plagued by large fluctuations, while foams are amenable to more conclusive experimental observations. Therefore, this mapping creates a formal gateway between the two systems and makes it possible to carry out experiment on one type of systems in order to learn about the other.

Acknowledgements:

It is a pleasure to acknowledge discussions with Prof R. C. Ball, Prof Sir S. F. Edwards and Dr D. V. Grinev. I am grateful to Prof M. E. Cates who stressed to me the importance of assuming the limit of infinitely thin cell walls to this analysis.

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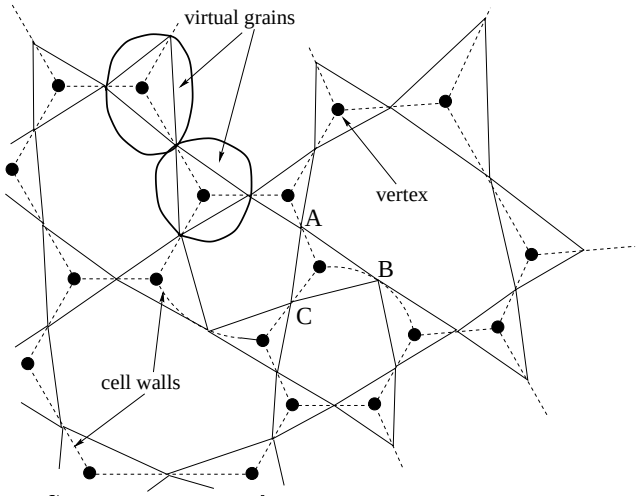

FIG. 4. The mapping from a trigonal cellular network onto an assembly of grains under external loading. The original cellular network is shown in dashed lines. The midpoints of the cell walls are joined to form triangles around each vertex (black circle). The grains of the equivalent granular assembly are centered on the vertices with the corners of the triangles representing contacts between neighboring grains. Two such virtual grains in contact are shown.

FIG. 5. The vectors $\vec{r}_{cv}$ and $\vec{\rho}_{cv}$ shared between vertex v and cell c. $\vec{r}_{cv}$ connects two neighboring wall midpoints and is one edge in a clockwise-directed triangle around vertex v. The vector $\vec{\rho}_{cv}$ points from the centroid of triangle v to the centroid of cell c.

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[25] The choice of the number of vertices N to be very large also minimises the effects of the boundaries on the following discussion due to the negligibly small boundary-to-volume ration. However, it can be shown [14] that even if these efects are taken into consideration the conclusions of the discussions remain unchanged

[26] It is true that to reach the stressed state the structure may have been strained from a different state, but the history of the system before arrival at the stressed state is irrelevant to the strut forces

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[28] This is a direct reult of applying Euler's theorem relating the number of cells, faces and edges to a planar structure. The theorem has been proven by Euler circa 1750

[29] In fact, this statement extends to all theories that use deformation-stress relationship as constitutive input to

solve for the stress field

[30] Although the normal forces are distributed all along the cell walls, it is possible to show that these can be uniquely represented by a localized force $F_{vv'}$ normal to the cell walls. Placing these forces at the midpoints of the walls is assumed here to introduce only a negligible error

[31] It should be noted that depending on the boundary loading, the stresses developing in the rigid frame formed by the $\vec{r}$-network would be either all compressive, or all tensile. For a proof of this statement see Maxwell, J.C., 1864. Phil. Mag., Ser 4, **27**, 250-261; Maxwell, J.C., 1864. Phil. Mag., **27**, 294-299; Maxwell, J.C., 1869. Trans. Roy. Soc. Edinb., **26**, 1-40